Now that Maxwell's equations for the radiated field have been developed a practical result is appropriate. When Maxwell's equations are written in terms of **E** and **B**, they result in eight simultaneous scalar first–order partial differential equations. In order to solve these equations in practice, the vector potential A is used.

Given a simple dipole antenna element, which is far removed from the ground and other objects consists of two straight wires with a generator connected between them.

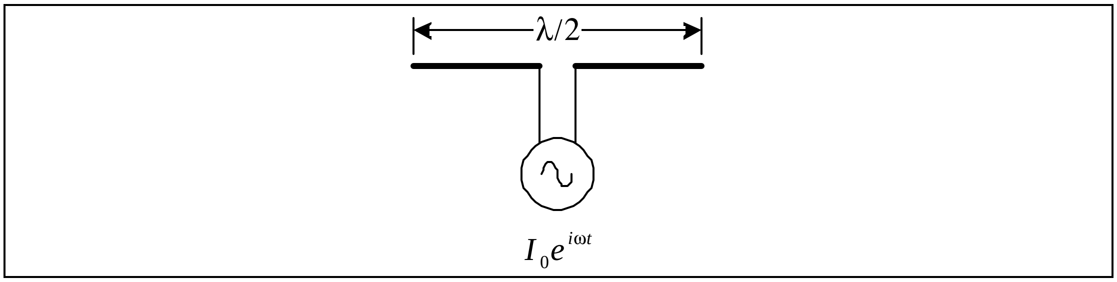

Figure 3.0 — Simple Dipole Antenna driven by the current source $I_0e^{i\omega t}$. The current distribution along the antenna's wires is known from the current sources time behavior. The radiated field produced by the flowing current follows the time behavior of the driving source.

If the current distribution along the antenna element is known, the far field radiation pattern can be found by integrating over the length of the antenna. Although the evaluation of the *simple dipole* may appear to be a straight forward problem, it is a very difficult boundary value problem even of the antenna wire is assumed to be a perfect conductor. For the antenna to radiate properly, the current must be zero at the ends of the wire where charges are deposited and the tangential electric field due to all currents and charges must vanish. In order to simplify further the evaluation of the dipole antenna it is assumed the current flowing in the antenna element varies sinusoidally with time as expressed by $I_0e^{i\omega t}$, where is the *phasor* current value. [1]

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[1] The *Phasor* representation is derived from the representation of time-varying fields through a Fourier series. Rather than use sinusoidal functions directly, it is convenient to introduce the complex exponential $e^{i\omega t}$. The advantage of this representation is that derivatives and integrals of $e^{i\omega t}$ are proportional to $e^{i\omega t}$, so that the function can be eliminated from all equations. Given a complex expression, $\rho = \rho_{real} + i\rho_{imag}$, the instantaneous value, as a function of time, of ρ is then given by the expression $\rho(t) = \mathrm{Re}[(\rho_{real} + i\rho_{imag})e^{iwt}] = \rho_{real}\cos\omega t - \rho_{imag}\sin\omega t$.

A simplified model of a dipole antenna will first be used. In this model the antenna of length l, will lie along the z–axis with the x–axis projected horizontally into free space. To further simplify the model the charge produced by the driving current is zero everywhere except at the ends of the antenna $z = \pm l/2$ where there is a time–dependent charge, $Q(t)$, produced by the changing antenna current, $I(t)$. This charge can be derived from the current as,

$$\frac{d}{dt} Q(t) = I(t) = I_0 \cos \omega_0 t \, , \tag{6.1}$$

which can be integrated to give,

$$Q(t) = \frac{1}{\omega_0} I_0 \sin \omega_0 t \, . \tag{6.2}$$

The *dipole moment* $\mathbf{p}$ is this charge expression times the physical separation of the charges. [2] The dipole moment is now given as,

$$\mathbf{p} = 2Q(t)\frac{l}{2}\mathbf{z} = \frac{I_0 l}{\omega_0} \sin \omega_0 t \mathbf{z} \, . \tag{6.3}$$

In order to further simplify the expression for the radiated field, the Lorentz gauge will be used in which the radiated field potentials satisfy,

$$\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = 0 \, . \tag{6.4}$$

The current density $\mathbf{j}$ in the antenna is given by,

$$\mathbf{j}(x) = \begin{cases} I_0 \cos \omega_0 t \delta(x) \delta(y), & -l/2 < \mathbf{z} < l/2 \\ 0, & \text{otherwise} \end{cases} \, , \tag{6.5}$$

and the vector potential is,

$$\mathbf{A}(x) = \frac{1}{c} \int \mathbf{j}(x') \frac{e^{ik|x-x'|}}{|x-x'|} d^3 x' \, . \tag{6.6}$$

If $x \gg l$ — far field radiation zone — then $|x - x'|$ can be approximated

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[2] In a more realistic description of the antenna, the current distribution along the antenna wire would not be constant and the charge density would be nonzero along the wire as well as the ends.

by $\mathbf{r} = \mathbf{n} \cdot x'$ where $\mathbf{n}$ is a unit vector along the x–axis and $|x| = \mathbf{r}$. Substituting this approximation into the field potential and rearranging the integral terms gives,

$$\mathbf{A}(x) \approx \frac{e^{ik\mathbf{r}}}{c} \int \mathbf{j}(x') e^{ik\mathbf{n}\cdot x'} d^3 x' . \tag{6.7}$$

Integrating over both x' and y' and letting $\mathbf{n} \cdot \mathbf{z} = \cos\theta$ where θ is the angle between $\mathbf{n}$ and the z–axis gives the vector potential as,

$$\mathbf{A}(x) \approx \mathbf{z} \frac{2I_0 e^{ik\mathbf{r}}}{\omega_0 \mathbf{r} \cos\theta} \sin\left(\frac{kl}{2} \cos\theta \right) \cos\omega_0 t . \tag{6.8}$$

For small values of kl this expression reduces to the far field approximation given in Eq. (6.8) as,

$$\mathbf{A}(x) = \frac{I_0 l e^{ik\mathbf{r}} \cos\omega_0 t}{c\mathbf{r}} . \tag{6.9}$$

By using spherical coordinates, this simplified example can be expanded further to illustrate some of the complexities of the *simple dipole*. The current element is in the z–direction with its location at the origin of a set of spherical coordinates. By evaluating the electric field potential given by Eq. (6.9) for any point Q at radius $\mathbf{r}$, in the z–direction,

$$\mathbf{A}_z = \frac{aI_0}{4\pi r} e^{-i(\ \omega r /\ \nu)} , \tag{6.10}$$

given the potential in spherical coordinates,

$$\mathbf{A}_r = \mathbf{A}_z \cos\theta = \frac{aI_0}{4\pi r} e^{-ikr} \cos\theta , \tag{6.11}$$

$$\mathbf{A}_\theta = -\mathbf{A}_z \sin\theta = -\frac{aI_0}{4\pi r} e^{-ikr} \sin\theta , \tag{6.12}$$

where $k = 2\pi/\lambda$.

There is no ϕ component of $\mathbf{A}$ and there are no variations with ϕ in any expressions because of the symmetry about the axis. The electric and magnetic field components may be found from Eq. (6.11) and Eq. (6.12) resulting in,

$$\mathbf{B}_\phi = \frac{aI_0}{4\pi r} e^{-ikr} \left(\frac{ik}{r} + \frac{1}{r^2} \right) \sin\theta , \tag{6.13}$$

$$\mathbf{E}_r = \frac{aI_0}{4\pi r} e^{-ikr} \left(\frac{2\eta}{r^2} + \frac{2}{i\omega r^3} \right) \cos\theta \,, \tag{6.14}$$

$$\mathbf{E}_\theta = \frac{aI_0}{4\pi r} e^{-ikr} \left(\frac{i\omega}{r} + \frac{1}{i\omega r^3} + \frac{\eta}{r^2} \right) \sin\theta \,, \tag{6.15}$$

where $\eta = \sqrt{\mu/\varepsilon} = 20\pi\,\Omega$ is the impedance of free space. [Sche52]

Evaluating the electric and magnetic field equations in the *radiation zone*, results in terms which vary as $1/r$.

$$\mathbf{B}_\phi = \frac{ikaI_0}{4\pi r} \sin\theta e^{-ikr} \,, \tag{6.16}$$

$$\mathbf{E}_\theta = \frac{i\omega aI_0}{4\pi r} \sin\theta e^{-ikr} = \eta \mathbf{B}_\phi \,. \tag{6.17}$$

Using the developments of §5, Eq. (6.17) exhibits the characteristics typical of uniform plane waves, with the Poynting vector in the radial direction,

$$\bar{S}_r = \frac{1}{2}(\mathbf{E}_\theta \times \mathbf{B}_\phi) = \frac{\eta k^2 I_0^2 a^2}{32\pi^2 r^2} \sin^2\theta, \text{ watts m}^{-2} \,. \tag{6.18}$$

The total power is given as the integral is the Poynting vector over any surrounding surface,

$$\begin{aligned} W &= \oint_S S \cdot dS = \int_0^\pi S_r 2\pi r^2 \sin\theta d\theta, \\ &= \frac{\eta k^2 I_0^2 a^2}{16\pi} \int_0^\pi \sin^3\theta d\theta, \\ &= \frac{\eta\pi I_0^2}{3} \left(\frac{a}{\lambda} \right)^2 \cong 40\pi^2 I_0^2 \left(\frac{a}{\lambda} \right)^2 \quad \text{watts.} \end{aligned} \tag{6.19}$$

§6.1. TIME-DEPENDENT FIELDS IN CONDUCTORS

The distinction between a conductor and a nonconductor of electricity was first made through experiments in the early eighteenth century. When an electric charge was applied to a nonconducting material the charge remained for some time. However when a charge was applied to a conducting material it rapidly spread over the body of the material. This *spreading* action is due to the mobility of charge–carrying particles. In conductors the charge–carrying are electrons or ions. In nonconducting materials or insulators the electrons are *bound* to atoms and only a very strong force can pull them away. [3]

Using this *simple* understanding of conduction an expression for the current in a conducting wire induced from the external electromagnetic field will be developed. The receipt of the electromagnetic wave by the conducting wire antenna will also be developed, to complete the communication path. This formulation will take a *simplified* approach to the description of the motion of the electrons in the conductor. A full description of the interaction of electromagnetic waves with a conductor is beyond the scope of this monograph, but can be found in [Pari69], [Jack62], [Eyge72].

The basic concept is that all matter contains charged particles — free electrons, bound electrons, ions, etc. When an electromagnetic wave *impinges* on a material the charged particles at set in motion, resulting in a spatial distribution of charge. These distributions of current will vary depending on the different macroscopic properties of the material. These behaviors can be summarized as:

> given certain approximations, a time harmonic wave of frequency ω will propagate in the material with an complex propagation constant k' that differs from the free space propagation constant $k = \omega/c$ [Eyge72], [Ramo84].

Materials react to applied electromagnetic fields in a variety of ways. For a metal or a semiconductor, where there are mobile electrons present, the electric polarization or *displacement current* given in Maxwell's Eq.(III) as well as the Lorentz force results in the motion of these electrons in the presence of a time varying electromagnetic field. [4]

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[3] It was Stephen Gray who first observed that there were two kinds of matter, one that could be given an electrical charge and one that could not. He also observed that some *electrified bodies* would repel each other while others attracted each other. The *attraction* was in direct conflict with the *effluvial* theories popular in Newton's time [Will66].

[4] In a dielectric the electrons are bound to the material's atoms. The a conductor the electrons are *free* to move in response to electromagnetic force. The number of conduction electrons varies with the conductor. In *noble metals* such as copper, silver and gold, the number of electrons is $\approx 10^{28}\, m^{-3}$. In semiconductors such as germanium the electron density is in the range of $10^{20}\, m^{-3}$ to $10^{24}\, m^{-3}$. In a weak plasma such as the ionosphere and *outer* space the electron density is $10^{6}\, m^{-3}$ to $10^{11}\, m^{-3}$.

A parameter in the propagation of electromagnetic waves is the *relaxation time* t between collisions of the charge carriers (electrons) which determines the conductivity and the frequency of the propagating waves in the conductor.

In most cases this motion can be treated as a *linear* response — proportional to the applied field. Often the response is also independent of the direction of the applied field, which allows the material to be considered isotropic. The response of these linear, isotropic materials to time–varying fields may also depend on the frequency of the field, where the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ are functions of frequency ω. Common dielectrics and conductors behave in this manner in normal applications.

In solids and liquids the permittivity differs significantly from the free space permittivity due to the behavior of the *bound electrons* in the materials atoms. In a conductor the movement of the free electrons in response to the applied electric field produces a current that overwhelms the electromagnetic field of the bound electrons [Ramo84].

§6.1.1. WAVE PROPAGATION IN A CONDUCTION MEDIA

A conductor placed in an oscillating electric field, $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, with a sufficiently low frequency of oscillation will produce an oscillating current density given by,

$$\mathbf{j}(\mathbf{r},t)=\sigma(\omega)\mathbf{E}(\mathbf{r},t), \tag{6.20}$$

where the frequency dependent *conductivity* is given by,

$$\sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}, \tag{6.21}$$

The expression is Eq. (6.20) is Ohm's Law. As a result a wave propagating in the conductor, in the z–direction, resulting from the external radiation is given by,

$$\mathbf{E}=\mathbf{E}_0 e^{-z/\delta}e^{i(z/\delta-\omega t)}, \tag{6.22}$$

and,

$$\mathbf{B}=\frac{(1-i)}{\omega\delta}z\times\mathbf{E}, \tag{6.23}$$

where,

$$\delta=\frac{c}{n\omega}=\sqrt{\left(\frac{2\varepsilon_0 c^2}{\sigma_0\omega}\right)}=\sqrt{\left(\frac{2}{\mu_0\sigma_0\omega}\right)}, \tag{6.24}$$

which is the *skin depth*.[5]

For *classical* materials (non–solid state devices) the motion of the charged particles in the material is determined by classical forces. Four different material states will be considered: a plasma, a conductor, a dielectric and a lossy dielectric.

In each material an electron of mass m and charge e is *driven* by an electric field, $\mathbf{E} = E_0 e^{-i\omega t}$. It will be assumed that the charge density is *dilute* enough so that no other forces are acting on the electrons. The electrons and surrounding positive charge form a *plasma* with the electron's equation of motion given as,

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[5] In most physics and engineering texts the skin depth term is usually given as a tautology with no further background. It can however be derived directly from Maxwell's equations. Starting with the homogeneous equations $1/\mu \nabla \times \mathbf{B} = \mathbf{j} + \varepsilon\, \partial \mathbf{E}/\partial t$ and $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and Ohm's Law $j = \sigma E$ a one dimensional example can be constructed. By using only one dimension Maxwell's equations for a particle moving in the z–direction can be written starting with $\partial/\partial x = \partial/\partial y = 0$ so that the current is moving only in the z–direction with $\nabla \times \mathbf{E} = \partial E_x/\partial z$ which gives $1 = (\omega\mu\sigma/2)^{1/2}\delta \to \delta = (2/\omega\mu\sigma)^{1/2}$ and $\nabla \times \mathbf{B} = -B_y/\partial z$ which gives $-\nabla B_y/\partial z = \mu\sigma E_x + \mu\varepsilon\, \partial E_x/\partial t$. These two Maxwell equations have a solution of the form $E_x = E_0 e^{-i(\omega t - \mathbf{k}z)}$ and $B_x = B_0 e^{-i(\omega t - \mathbf{k}z)}$. By defining the terms $\partial/\partial z = -i\mathbf{k}$ and $\partial/\partial t = -i\omega$ the differential equations can be changed to the algebraic equations $i\mathbf{k}E_x = i\omega B_y$ and $-i\mathbf{k}B_y = (\mu\sigma - i\omega\mu\varepsilon)E_x$. This homogeneous system of equations can be solved if the determinant of the coefficients vanishes such that $\begin{vmatrix} -i\mathbf{k} & i\omega \\ \mu\sigma - i\omega\mu\varepsilon & i\mathbf{k} \end{vmatrix} = 0$. Expanding the determinant gives $k^2 - i\omega(\mu\sigma - i\omega\mu\varepsilon) = 0$. This expression is usually called the *dispersion* equation because it described the relationship between the frequency ω and the wave number $\mathbf{k}$, which is related to the phase velocity of the wave $v_{phase} = \omega/\mathbf{k}$. Unless ω and $\mathbf{k}$ are linearly related different frequencies will propagate at different velocities. A medium in which the conductivity is zero $\sigma = 0$ and μ and ε are independent of frequency is nondispersive. The relationships between $\mathbf{k}$ and ω is given by $k = \omega\sqrt{\mu\varepsilon}$. Solving the dispersion equation gives $\mathbf{k} = (\omega^2\mu\varepsilon + i\omega\mu\sigma)^{1/2}$. Whenever the conductivity is nonzero, the wavenumber is complex. The spatially varying part of the wave equation is then $e^{i\mathbf{k}z} = e^{i(k_{real} + ik_{imag})z} = e^{ik_{real}z}e^{-ik_{imag}z}$. If the imaginary part of $\mathbf{k}$ is positive the amplitude of the wave declines exponentially. If the conductivity of the medium is large, the second term of the dispersion equation dominates giving $k \approx (i\omega\mu\sigma)^{1/2} = \{\pm(i+1)/\sqrt{2}\}\{(\omega\mu\sigma)^{1/2}\}$. The rate at which the electromagnetic field decays in a good conductor is given by this expression. Since $k_{imag} = (\omega\mu\sigma/2)^{1/2}$, the electric field amplitude varies as $|E_x| = E_0 e^{-(\omega\mu\sigma/2)^{1/2} z}$. The distance at which the amplitude decays to $1/e$ is defined to be the skin depth and is given by $1 = (\omega\mu\sigma/2)^{1/2}\delta \to \delta = (2/\omega\mu\sigma)^{1/2}$. [Soly93], [Zima72], [Jack75], [Pano55].

$$m\ddot{\mathbf{r}} = eE_0 e^{ik\cdot\mathbf{r}-i\omega t}\,. \tag{6.25}$$

The *ionosphere* is an example of such a *free electron* material.

In a *conductor* there is a force applied to the *free* electron from the collisions it makes with the impurities or with the vibrating lattice of ions in the material. These collision forces can be described as a *damping force*, γ, proportional to the velocity of the electron ν and is written as $m\gamma\nu$. The equation of motion of an electron in a conductor becomes,

$$m\ddot{\mathbf{r}} + m\gamma\,\dot{\mathbf{r}} = eE_0 e^{ik\cdot\mathbf{r}-i\omega t}\,, \tag{6.26}$$

which is the equation of motion for a *damped* oscillator.

For a *pure dielectric*, there is no damping but rather the *free* electrons are now bound to an origin with a natural frequency. This frequency is a *crude* counterpart of the natural frequencies of electrons bound in atoms. In this model the electrons experience a *restoring* force which involves this frequency given by, $-m\omega_0^2\mathbf{r}$, which gives the electrons equations of motion as,

$$m\ddot{\mathbf{r}} + m\omega_0^2\mathbf{r} = eE_0 e^{ik\cdot\mathbf{r}-i\omega t}\,. \tag{6.27}$$

The final model of a material is a *lossy dielectric* in which the equations of motion involve both a *damping* term and a harmonic *restoring* force, which gives the equations of motion as,

$$m\ddot{\mathbf{r}} + m\gamma\,\dot{\mathbf{r}} + m\omega_0^2\mathbf{r} = eE_0 e^{ik\cdot\mathbf{r}-i\omega t}\,. \tag{6.28}$$

In the lossy dielectric and dielectric model any displacement of the dilute charge density from the central ion force produces a restoring force [Hipp54]. This restoring force interacts with the inertia of the moving charge *cloud* to produce a resonance similar to a mass–spring mechanical system. The displacement of one ion from another also produces a resonance in the ionic polarizability. This resonance however appears at lower frequencies than the purely *electronic* contribution because of the larger masses of the individual ions [Ramo84].

There are also losses or *damping* in each of the resonances rising from radiation from the free electrons. The motion of the electrons described in the last four equations produces a *point current*. In order to consider the effects of this current the steady–state solution to the equations of motion must be found.

If the equation of motion for a lossy dielectric is considered the most general of the four models, the steady–state solution can be found by solving for the position and velocity variables, using the complex notion of $\mathbf{r} = \mathbf{r}_{real} e^{-i\omega t}$, which gives,

$$\mathbf{r}_{real} = \frac{eE_0}{m\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)} . \tag{6.29}$$

From the relation $\nu_{real} = -i\omega\mathbf{r}_{real}$, the velocity becomes,

$$\nu_{real} = \frac{-i\omega eE_0}{m\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)} . \tag{6.30}$$

The equations of motion for a plasma, conductor or dielectric can now be found by setting γ or ω_0 or both to zero.

In order to properly *explain* the motion of charges in materials an extensive development of the underlying theory of solids is necessary. As an alternative a *heuristic* approach will be taken using the description of electrons and their motion which results in a current, that has been presented above. In this heuristic description many electrons are put in motion by an external electric field. The resulting motion produces a current $\mathbf{j}$ based on the number electrons per unit volume passing through a surface *N*, such that the *average* current is given by,

$$\mathbf{j} = Ne\nu . \tag{6.31}$$

There are several *assumptions* made here that simplify the description, but would require more complex descriptions in order to properly explain actual phenomenon.

Using the expression for the velocity of an individual electron, the conductivity constant σ can be given as,

$$\sigma = -\frac{iN\omega e^2}{m\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)} . \tag{6.32}$$

In conductors the *natural frequency* ω_0 is zero so that,

$$\sigma = -\frac{iN\omega e^2}{m\left(\omega^2 - i\omega\gamma\right)} . \tag{6.33}$$

It has been experimentally that the damping constant γ is of the order $10^{17}\,\text{sec}^{-1}$, which has the dimensions of frequency. This means that for electrons *driven* at frequencies less than this, ω^2 in the denominator is negligible and the conductivity constant is given approximately as,

$$\sigma = \frac{e^2 N}{m\omega\gamma}. \tag{6.34}$$

Which allows a crude derivation of Ohm's Law to be restated as,

$$\mathbf{j} = \sigma E, \tag{6.35}$$

which embodies *Drude's model of conductivity* [Egye72] [6]. The complex propagation constant k' is now given as,

$$k'^2 = \mu\varepsilon\frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right). \tag{6.36}$$

The first term corresponds to the displacement current and the second term to the conduction current. [7] Assuming σ, μ and ε are real the propagation constant can be simplified to,

$$k = \beta + i\frac{\alpha}{2},$$

where the real part is given by,

$$\beta = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\sqrt{\frac{\left(\sqrt{1+\left(4\pi\sigma/\omega\varepsilon\right)^2}+1\right)}{2}}, \tag{6.37}$$

and the imaginary part is given by,

$$\frac{\alpha}{2} = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}\sqrt{\frac{\left(\sqrt{1+\left(4\pi\sigma/\omega\varepsilon\right)^2}-1\right)}{2}}. \tag{6.38}$$

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[6] This simple model of free electron conductivity was first constructed by Paul Karl Ludwig Drude (1863–1906), in 1900.

[7] One of the notational complexities of electromagnetic theory is the definitions of the various constants and propagation vectors. It seems that each author of the various bibliographical sources has chosen a *different* notation. In this monograph the following notation is used in the context of classical electrodynamics. The italic letter *k* represents the propagation constant and the bold faced letter **r** represents the propagation vector. In the quantum mechanical description of electrodynamics the bold face letter **k** represents the propagation vector and the propagation constant, which contains information regarding the medium's characteristics is absent from the expression.

The exponential form of the propagation vector k now becomes,

$$e^{i(k\cdot\mathbf{r}-\omega t)} \equiv e^{-\alpha/2-\omega t} e^{-\beta\cdot\mathbf{r}-\omega t} \tag{6.39}$$

In metals σ is again $10^{17}\,\sec^{-1}$ which means that $4\pi\sigma\omega/c^2$ is much larger than ω^2/c^2 which allows the *skin depth* to be restated as,

$$\delta = \frac{c}{\sqrt{2\pi\mu\sigma\omega}}, \tag{6.40}$$

using the relation $\sqrt{i} = (1+i)/\sqrt{2}$ and $k' = (1+i)/\delta$.

The electric and magnetic fields *propagating* in the metallic conductor are now given as,

$$\left.\begin{aligned} \nabla^2\mathbf{E} + k'^2\mathbf{E} = 0, \\ \nabla^2\mathbf{B} + k'^2\mathbf{B} = 0. \end{aligned}\right\} \tag{6.41}$$

§6.2. ELECTROMAGNETIC WAVES INCIDENT ON A CONDUCTOR

The study of uniform plane waves propagating in a region which has an abrupt change in conductivity, permittivity and permeability is referred to as the study of *reflection* and *refraction*. The analysis of the effects on the plane wave as it crosses various media boundaries is complex. Several generalizations will be used, as in the previous section, to illustrate the principles without undue detail. [8]

Suppose that an electromagnetic wave is propagating in one medium (1) and encounters a *discontinuity* of a second medium (2). If the dimensions of the second medium are large compared to the wavelength of the propagating wave a fraction of the incident energy will be *reflected* from the surface of medium (2) and the remaining energy will be *transmitted* into the second medium. The direction the electromagnetic energy takes in the second medium will be different from the original direction in medium (1). This change in direction is referred to as

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[8] The purpose of this portion of the monograph is to provide the reader with a *background* in the issues of reflection and refraction of electromagnetic waves. One *major* simplification is done by ignoring the effects of the orientation of the electric vector on the reflection and refraction of the incident wave. Two orientations are possible, the **E** vector normal to the plane of incidence and the **E** vector in the plane of incidence. For a detailed discussion of the complex subject the reader is referred to [Jack75], [Lorr70] whose source material was taken from the classic text on the subject of electromagnetism [Stra41].

refraction. The laws governing reflection and refraction of electromagnetic waves at the surfaces of an infinite dimension discontinuity are relatively simple [Eyge72], [Jack75], [Beck64]. If the dimensions of the discontinuity are on the order of the wave length of the radiation, the mathematical description of the wave's behavior becomes difficult [Jack75], [Stra41]. The disturbance to the propagating electromagnetic field in this case is called *diffraction*. In both cases — refraction and diffraction — the electromagnetic field in medium (2) induces the conducting charges into motion. Both situations are classified as *inhomogeneous boundary–value* problems. [9]

A situation is that of a conductor embedded in a dielectric medium. Charges in medium (2) are displaced from their equilibrium distribution of the surface of the conductor. The resulting oscillations produce oscillations in the surrounding field. This *induced* field can be represented as a *superposition* of the characteristic waves functions whose form is determined by the configuration of the conducting body and whose relative amplitudes fixed by the initial conditions [Stra41]. This is called a *homogeneous boundary–value* problem.

Associated with each characteristic of the conducting medium is a *characteristic number* that determines the frequency of that particular oscillation. The oscillation are damped, partly due to the finite conductivity of the material and partly because the energy dissipated in radiation. In this situation the positions of the conductor and the dielectric relative to the surface of separation can be interchanged. The electromagnetic oscillations then take place within a dielectric *cavity* formed by the boundaries of the conducting material [Stra41].

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[9] The origins of the theory of reflection and refraction can be traced to Newton's *Opticks* [Cohe52]. In *Opticks* Newton attempted to describe light rays as *very small Bodies emitted from shining Substances*. Newton postulated that these Bodies…

> … will pass through uniform Mediums in right Lines without bending into the Shadow, which is the Nature of the Rays of Light.

In Newton's theory light was a particulate of nature just as ordinary matter.

In order to explain reflection and refraction Newton theorized that the *corpuscles* of light were repelled by…

> …some power of the Body, which is evenly diffused over the surface and by which it acts upon the Ray without immediate Contact. The power could repel the corpuscles of lights during the reflection and pull them through …

…the glass during refraction.

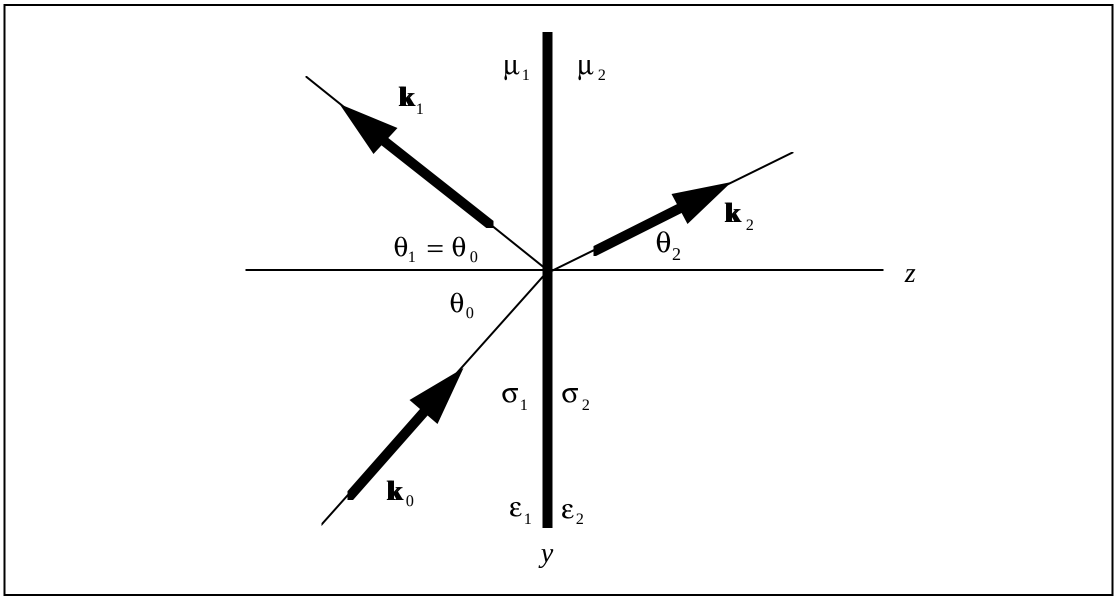
The general law of reflection and refraction describing the behavior of plane waves between two linear, homogeneous, isotropic media is called *Snell's Law*[10] and is given as,

$$\frac{\sin\theta_i}{\sin\theta_r}=\frac{v_i}{v_r}, \tag{6.42}$$

where the angle if incidence of the incoming plane wave is θ_i and the angle of refraction of the incoming plane wave is θ_r and v_i and v_r is the phase velocities of the plane wave in the medium of incidence and refraction.

In geometric optics these laws were derived through intuitive observations, but they can also be derived by using the Maxwell equations to evaluate *Transverse Electromagnetic Waves* (TEM) [Pari69].

For an electromagnetic wave with a propagation vector $\mathbf{r}_1$ incident on a conductor with conductivity σ_2, **Figure 4.0** describes the reflection and refraction effects.



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[10] This law was apparently discovered by Willebrord Snell around 1621. However Snell never published his result which lead Descartes to attempt to take credit for Snell's work in his *Dioptrique* in 1637. Descartes derivation was deductive and incorrect. It was Fermat that first derived the correct formulation in 1661, using the Principal of Least Time. Since Fermat's principal was based on metaphysical rather than physical reasoning it had no lasting influence. Hamilton provided the physical foundation for Snell's Law using his variational principals in 1834. See §8 for the development of Hamilton's approach to mechanics.

Figure 4.0 — Spatial relationships for the description of reflection and refraction at a dielectric–conductor boundary. An *incoming* plane wave with a propagation vector $\mathbf{k}_0$ is incident on a boundary at $z = 0$. A *reflected* wave $\mathbf{k}_1$ and a transmitted wave $\mathbf{k}_2$ result from the incoming wave.

The incident wave can be written using the exponential notation as,

$$\mathbf{E}_i = \hat{y} E_{i0} e^{i(\omega t - k_0 \cdot \mathbf{r}_0)}, \tag{6.43}$$

and,

$$\mathbf{B}_i = \hat{z} B_{i0} e^{i(\omega t - k_0 \cdot \mathbf{r}_0)}. \tag{6.44}$$

where $\hat{z}$ and $\hat{y}$ are unit vectors in the z and y direction. The *reflected* wave cane be written in a similar manner as,

$$\mathbf{E}_r = \hat{y} E_{r0} e^{i(\omega t - k_1 \cdot \mathbf{r}_1)}, \tag{6.45}$$

and,

$$\mathbf{B}_r = \hat{z} B_{r0} e^{i(\omega t - k_1 \cdot \mathbf{r}_1)}. \tag{6.46}$$

and the *transmitted* wave as,

$$\mathbf{E}_t = \hat{y} E_{t0} e^{i(\omega t - k_2 \cdot \mathbf{r}_2)}, \tag{6.47}$$

and,

$$\mathbf{B}_t = \hat{z} B_{t0} e^{i(\omega t - k_2 \cdot \mathbf{r}_2)}. \tag{6.48}$$

The propagation constants for the wave traveling in the refracted medium is given by,

$$k_1 = i\beta_1 = i\omega\sqrt{\mu_1\varepsilon_1}, \tag{6.49}$$

and wave traveling in the transmitted medium can now be given as,

$$k_2 = \alpha_2 + i\beta_1, \tag{6.50}$$

where,

$$\beta_2 = \sqrt{\mu_2\varepsilon_2}\left(\frac{\omega}{c}\right)\sqrt{\frac{\left(\sqrt{1+\left(4\pi\sigma_2/\omega\varepsilon_2\right)^2}+1\right)}{2}}, \tag{6.51}$$

and,

$$\frac{\alpha_2}{2} = \sqrt{\mu_2 \varepsilon_2}\left(\frac{\omega}{c}\right)\sqrt{\frac{\left(\sqrt{1+\left(4\pi\sigma_2/\omega\varepsilon_2\right)^2}-1\right)}{2}}\,. \tag{6.52}$$

Within the conducting medium the transmitted wave is now given by,

$$\mathbf{E}_t = \mathbf{E}_2 e^{-k_2(\mathbf{k}_2\cdot\mathbf{r})+i\omega t} = \mathbf{E}_2 e^{-k_2(y\sin\theta_2 - z\cos\theta_2)+i\omega t}\,. \tag{6.53}$$

The *transmitted* waves propagation constant is complex and requires further expansion. Using Snell's Law

$$\sin\theta_2 = \frac{k_1}{k_2}\sin\theta_0 = \frac{i\beta_1}{\alpha_2 + i\beta_2}\sin\theta_1\,, \tag{6.54}$$

resulting in,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\left(\frac{i\beta_1}{\alpha_2 + i\beta 2}\right)^2 \sin^2\theta_1}\,. \tag{6.55}$$

It is convenient to set,

$$\cos\theta_2 = \mathbf{r}\cdot e^{i\delta} \tag{6.56}$$

in terms of the spatial component of Eq. (195) becomes,

$$\begin{aligned} k_2\left(y\sin\theta_2 - z\cos\theta_2\right) &= \left(\alpha_2 + i\beta_2\right)\left(y\frac{i\beta_1}{\alpha_2 + i\beta_2}\sin\theta_1 - z\mathbf{r}e^{i\delta}\right), \\ &= i\left(\beta_1\sin\theta_1\right)y - \mathbf{r}\left(\alpha_2\cos\delta - \beta_2\sin\delta\right)z - \\ &\quad -i\mathbf{r}\left(\alpha_2\sin\delta - \beta_2\cos\delta\right), \\ &= -pz + i\left[\left(\beta_1\sin\theta_1\right)y - qz\right]. \end{aligned} \tag{6.57}$$

where,

$$p = \mathbf{r}\left(\alpha_2\cos\delta - \beta_2\sin\delta\right), \tag{6.58}$$

and,

$$q = \mathbf{r}\left(\alpha_2\sin\delta - \beta_2\cos\delta\right), \tag{6.59}$$

are real quantities, dependent on the parameters of the medium and the angle of incidence of the incoming wave, θ_1.

Substituting these expressions into Eq. (195) gives an expression for the *transmitted* wave as,

$$\mathbf{E}_t = \mathbf{E}_2 e^{-pz + i\left[\left(\beta_1 \sin\theta_1\right)y - qz + \omega t\right]}. \tag{6.60}$$

This refracted wave has planes of constant amplitude parallel to the boundary of the plane, $z = \text{constant}$, and has planes of constant phase inclined to its normal $\mathbf{k}$, $\left(\beta_1 \sin\theta_1\right)y - qz = \text{constant}$, at an angle which is no longer equal to θ_2, as shown in **Figure 5.0**.

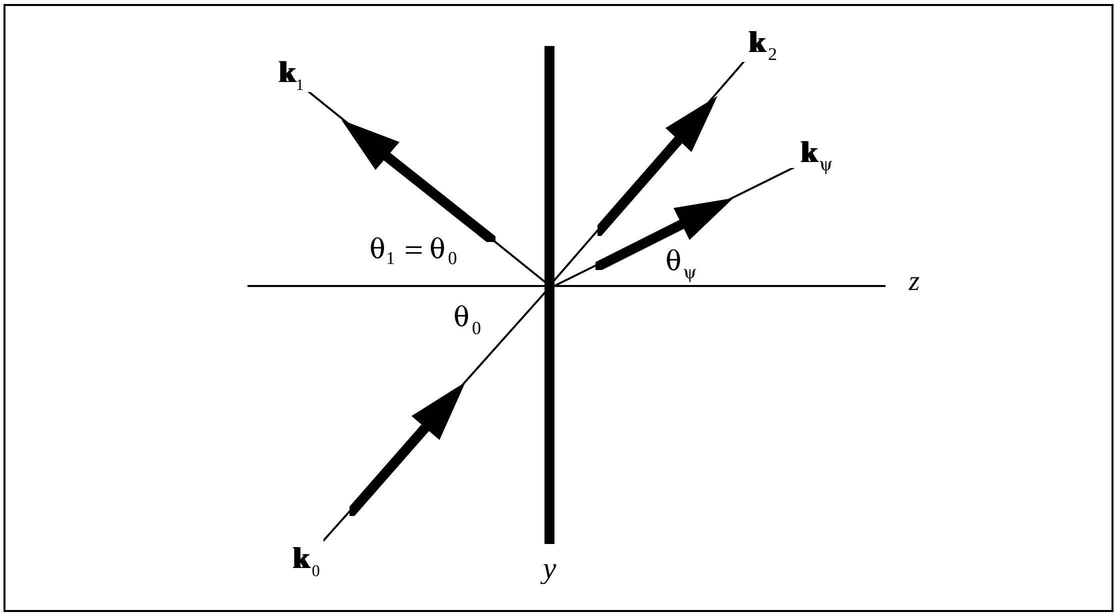

Figure 5.0 — Spatial relationships for the description of reflection and refraction at a dielectric–conductor boundary, with the *refracted* wave $\mathbf{k}_\psi$ which is different from the *transmitted* wave $\mathbf{k}_2$.

The true angle of refraction θ_ψ is given by the imaginary component of Eq. (72) as,

$$\begin{aligned}&\left(\beta_1 \sin\theta_1\right)y - qz + \omega t = \\ &-\sqrt{\left(\beta_1 \sin\theta_1\right)^2 + q^2} \times \left[\frac{\beta_1 \sin\theta_1}{\sqrt{\left(\beta_1 \sin\theta_1\right)^2 + q^2}}(-y) + \frac{qz}{\sqrt{\left(\beta_1 \sin\theta_1\right)^2 + q^2}}\right] + \omega t\end{aligned} \tag{6.61}$$

By defining two terms,

$$\sin\theta_\psi = \frac{\beta_1 \sin\theta_1}{\sqrt{\left(\beta_1 \sin\theta_1\right)^2 + q^2}}, \tag{6.62}$$

and,

$$\cos\theta_\psi = \frac{q}{\sqrt{(\beta_1 \sin\theta_1)^2 + q^2}} . \tag{6.63}$$

Eq. (203) can be rewritten as,

$$-\sqrt{(\beta_1 \sin\theta_1)^2 + q^2}\left(-y\sin\theta_\psi + z\cos\theta_\psi\right) + \omega t = -\sqrt{(\beta_1 \sin\theta_1)^2 + q^2}\,\mathbf{k}_\psi \cdot \mathbf{r} + \omega t , \tag{6.64}$$

where,

$$\mathbf{k}_\psi = -\sin\theta_\psi + \cos\theta_\psi . \tag{6.65}$$

Evaluating for the *refraction* angle θ_ψ gives,

$$\theta_\psi = \tan^{-1}\frac{\beta_1 \sin\theta_1}{q} . \tag{6.66}$$

In a *good conductor* where $\sigma_2/\omega\varepsilon_2 \gg 1$,

$$\alpha_2 \approx \beta_2 \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}} . \tag{6.67}$$

With the condition that σ_2 is large in a conductor and $\mu_1 \approx \mu_2$ Snell's law results in,

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{i\omega\sqrt{\mu_1\varepsilon_1}}{\sqrt{\omega\mu_2\sigma_2/2}\,(1+i)} = \sqrt{\frac{\omega\varepsilon_1}{\sigma_2}}\,e^{i\pi/4} \to 0 . \tag{6.68}$$

With this result,

$$p \approx \alpha_2 \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}} , \tag{6.69}$$

and,

$$q \approx \beta_2 \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}} , \tag{6.70}$$

and,

$$\theta_\psi = \tan^{-1} \frac{\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1}{\sqrt{\omega\mu_2\sigma_2/2}} \to 0\,. \tag{6.71}$$

In the incident medium, when the conductivity increases, the true angle of refraction tends to zero $\theta_\psi \to 0$, and the planes of constant phase are oriented parallel to the reflecting plane and to the planes of constant amplitude. The result is that in the receiving antenna, the incident electromagnetic wave generates a propagated wave of the form,

$$\mathbf{E}_t = \mathbf{E}_2 e^{-\alpha_2 z + i\left[\left(\beta_1 \sin\theta_1\right) y + \beta_2 z + \omega t\right]}\,. \tag{6.72}$$

§6.3. SUMMARY OF MAXWELL'S CLASSICAL FIELD THEORY

By the end of the 19th century Maxwell's four equations described all known electric and magnetic phenomena, which in turn could be used to describe all *macro* behaviors in chemistry and biology. The equations Maxwell formulated in 1861 are still valid today, it is their interpretation that has changed. The next sections will lay the foundation for the modern theory of electromagnetic radiation — Quantum Electrodynamics (QED). In this theory the electromagnetic force is carried through space by the *quanta* of the electromagnetic field — the photon. It is at this point the classical description of the electromagnetic field must be extended to include quantum mechanics and special relativity. In this monograph the formulation of this theory starts with the classical description of a charged particle moving in a potential field. In this way, the Lagrangian and Hamiltonian dynamics can be developed, then *quantized* and expanded to the 4–dimensions of special relativity.

> *Curiouser and curiouser, cried Alice...*
>
> — Through the Looking Glass [Dodg60]