§9. LAGRANGIAN OF THE ELECTROMAGNETIC FIELD

In the previous section the Lagrangian and Hamiltonian of an ensemble of point particles was developed. This approach is based on a discrete set of coordinates $q_i(t)$. This discrete formulation can be extended to a continuous formulation. Each point in a region of space – either finite or infinite will be associated with a continuous variable,

$$\phi(x,t). \tag{9.1}$$

The set of variables constitute a system with an infinite number of degrees of freedom – a field. In this formulation, the Lagrangian of the field variable is *functional* of the field. A functional is a mapping from a space of functions, the Lagrangians, to a set of real numbers. This mapping is given by,

$$L(t) = L\left[\phi(x,t),\dot{\phi}(x,t)\right]. \tag{9.2}$$

The functional $L(t)$, depends on the value of ϕ and $\dot{\phi}$ at all points in space at simultaneous time. Hamilton's principal can now be applied to this functional by defining a variation of the functional as,

$$\begin{aligned}\delta F[\phi] &= F[\phi+\delta\phi]-F[\phi],\\ &\equiv \int \frac{\delta F[\phi]}{\delta\phi(x)}\delta\phi(x)d^3x.\end{aligned} \tag{9.3}$$

The term in Eq. (9.3) $\delta F[\phi]/\delta\phi(x)$ is called the functional derivative of the functional $F[\phi]$ with respect to the functional ϕ at a spatial point x. This derivative decribes how the functional is changing when the values of the function ϕ is varied at the point x.

The functional derivative is formally defines as,

$$\frac{\delta F}{\delta g(y)} = \lim_{\varepsilon\to 0}\frac{F\left[g(x)+\varepsilon\delta(x-y)\right]-F\left[g(x)\right]}{\varepsilon} \tag{9.4}$$

where $\delta(x-y)$ is the Dirac delta function. The derivative in Eq. (9.4) displays all the properties of a standard derivative. [1]

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[1] If F and G are two functionals, than their product is a functional derivative defined as, $\delta[FG]/\delta g(x) = G(\delta F/\delta g(x)) + F(\delta G/\delta(x))$, which is the Leibniz property. If $F[g]$ is a functional that is well behaved in the interval in the function space around $g=0$, than

The functional derivative can now be applied to the Lagrangian in Eq. (9.3) to give,

$$\delta L\left[\phi,\dot{\phi}\right]=\int\left(\frac{\delta L}{\delta\phi(x)}\delta\phi(x)+\frac{\delta L}{\delta\dot{\phi}(x)}\delta\dot{\phi}(x)\right)d^3x \qquad (9.5)$$

The Lagrangian can now be integrated to produce the action function, usually denoted by $W\left[\phi,\dot{\phi}\right]$, which is a functional of ϕ and $\dot{\phi}$. By integrating over the time interval $t_1=-\infty$ to $t_2=+\infty$ to give,

$$\begin{aligned}\delta W &=\delta\int_{t_1}^{t_2} L\left[\phi,\dot{\phi}\right]dt,\\ &=\int_{t_1}^{t_2}\left(\frac{\delta L}{\delta\phi(x,t)}\delta\phi(x,t)+\frac{\delta L}{\delta\dot{\phi}(x,t)}\delta\dot{\phi}(x,t)\right)dt,dx^3, \qquad (9.6)\\ &=\int_{t_1}^{t_2}\left(\frac{\delta L}{\delta\phi(x,t)}-\frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}(x,t)}\right)\delta\phi(x,t)\,dtdx^3.\end{aligned}$$

Since Hamilton's stationary function is,

$$\delta W\left[\phi,\dot{\phi}\right]=\delta\int_{t_1}^{t_2} L\left(\phi,\dot{\phi}\right)dt=0, \qquad (9.7)$$

the Euler–Lagrangian can be generalized to Classical Field theory as,

$$\frac{\partial L}{\partial\phi}-\frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}}=0\,. \qquad (9.8)$$

Applying the principals developed in §8 to Maxwell's equations began as early as the 1870's. Maxwell made use of Hamilton's principal in his *Treatise* [Buch85]. By formulating the electromagnetic field as a continuous dynamic medium, Maxwell built on the ideas put forth by William Thomson and Peter Tait in [Thom62]. Thomson and Tait considered Hamilton's principal as a fundamental formulation of physics…

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the functional F can be expanded in a Taylor series $F[g]=\sum 1/n!\int dx_1, dx_2,\dots, dx_n\left(\delta^n F[g]\right)\Big/\left(\delta g(x_1)\cdots\delta g(x_n)\right)\Big|_{g=0}$.

> …(the) principal of *Least Action* is a useful guide to kinetic investigations. We are strongly impressed with the conviction that a much more profound importance will be attached to it … in the theory of several branches of physical science now beginning to receive dynamical explanation … his method of *Varying Action* which undoubtedly become a most valuable aid in the further generalization [Thom62] Vol2. §326.

Restating Hamilton's principal –– requires that the path taken by a physical system between two states at specified times and with fixed values of the variables at these times such that the value of the function $\int (T - V)\,dt$, where T is the kinetic energy V is the potential energy, must be an extermimum such that,

$$\delta \int_{t_0}^{t_1} (T - V)\,dt = 0\,.$$

In this form Hamilton's principal is capable of generating both the equations of motion of the electrodynamic system and the boundary conditions for any continuous field with localized forms of energy. By adopting Hamilton's principal as the fundamental formulation of electromagnetics, then every problem in field theory reduces to finding an appropriate expression for the field's potential and kinetic energies. [2] The essence of the Lagrangian or Hamiltonian approach consists of two parts: (i) the specification of the generalized coordinates which fix the state of the field and (ii) a choice of expressions, in terms of these coordinates and their spatial and temporal derivatives.

To make use of the results developed in §8, the Lagrangian, L, for the electromagnetic field *in vacuum* will be constructed. The f_r's are given by the vector and scalar potentials and the field equations take the form,

$$\left.\begin{aligned} \nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} &= 0 \\ \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} &= 0 \end{aligned}\right\} .(9.9)$$

The potentials have to be subjected to the further constraint of the Lorentz condition,

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[2] As is always the case this simplification of the situation is not universally true. Independent conditions on the boundary values or generalized coordinates must be imposed where the fields energies are functions.

$$\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = 0 . \tag{9.10}$$

The Lagrangian which provides the equations in Eq. (8.X) is given by,

$$\mathsf{L} = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right) .^{[3]} \tag{9.11}$$

The **B** and **E** fields can now be expressed in terms of the potentials, **A** and ϕ as,

$$\mathsf{L} = \frac{1}{8\pi}\left\{\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right)^2 - \left(\nabla \times \mathbf{A}\right)^2\right\} . \tag{9.12}$$

Expanding Eq. (9.11) in Cartesian coordinates,

$$\begin{aligned} \mathsf{L} = \frac{1}{8\pi}\Bigg\{ & \left(-\frac{\partial \phi}{\partial q_1} - \frac{\partial A_1}{\partial t}\right)^2 + \left(-\frac{\partial \phi}{\partial q_2} - \frac{\partial A_2}{\partial t}\right)^2 + \left(-\frac{\partial \phi}{\partial q_3} - \frac{\partial A_3}{\partial t}\right)^2 - \cdots \\ & \cdots - \left(\frac{\partial A_3}{\partial q_2} - \frac{\partial A_3}{\partial q_3}\right)^2 - \left(\frac{\partial A_1}{\partial q_3} - \frac{\partial A_1}{\partial q_1}\right)^2 - \left(\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}\right)^2 \Bigg\} . \end{aligned} \tag{9.13}$$

Using the Lagrange relationship,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathsf{L}}{\partial\left(\dfrac{\partial A_1}{\partial t}\right)}\right) + \sum_r \left\{\frac{\partial}{\partial q_r}\left(\frac{\partial \mathsf{L}}{\partial\left(\dfrac{\partial A_1}{\partial t}\right)}\right)\right\} - \frac{\partial \mathsf{L}}{\partial A_1} , \tag{9.14}$$

gives for the A_1 component of the vector potential,

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[3] The factor $1/8\pi$ is completely arbitrary and is used so in the integration of the electromagnetic field for charged bodies. There are other changes in the notation used in this section, including the absence of the absolute value signs around the **E** and **B** field variables — it will be assumed that the *real* nature of these variables is understood by the reader. In all instances the field variables are assumed to be functions of time and space — although no explicit parameters are shown in the equations.

Where there is a explicit dependence on a position variable a subscript will be used to indicate the dependent variable. In general the variable q_r will be used for a generalized coordinate value in place of **r** or *x*.

$$\begin{aligned}
\mathsf{L} &= \frac{1}{4\pi}\left\{\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial q_1}+\frac{\partial A_1}{\partial t}\right)+\frac{\partial}{\partial q_2}\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)-\frac{\partial}{\partial q_3}\left(\frac{\partial A_1}{\partial q_3}-\frac{\partial A_3}{\partial q_1}\right)\right\}^2, \\
&= \frac{1}{4\pi}\left\{-\nabla^2 A_1+\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial}{\partial q_1}\left(\nabla\cdot\mathbf{A}+\frac{\partial \phi}{\partial t}\right)\right\}, \\
&= -\frac{1}{4\pi}\left(\nabla^2 A_1-\frac{\partial^2 A_1}{\partial t^2}\right).
\end{aligned} \tag{9.15}$$

with similar equations for A_2 and A_3 components.

A similar expansion can be developed for the scalar potential, such that,

$$\mathsf{L} = -\frac{1}{4\pi}\left(\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}\right). \tag{9.16}$$

A Lagrangian which includes the Lorentz condition and represents the electromagnetic field independent of any constraints is given by,

$$\mathsf{L} = \frac{1}{8\pi}\left\{\left(-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}\right)^2-(\nabla\times\mathbf{A})^2-\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)^2\right\}. \tag{9.17}$$

Considering the Lagrangian for a electromagnetic field, with charge sources, is the next step in the development of the complete Hamiltonian.

The potential equations as originally developed in Eq. (4.11) are,

$$\left.\begin{aligned}
\nabla^2\mathbf{A}-\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mathbf{j} \\
\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2} &= -\rho
\end{aligned}\right\}. \tag{9.18}$$

The Lagrangian that will result in these equations is Eq. (9.17) plus an additional term $\rho(\mathbf{A}\cdot v-\phi)$, where ρ is the charge density and v is the velocity of the charge, so that the radiation and interaction Lagrangian is given by,

$$\mathsf{L} = \frac{1}{8\pi}\left\{\left(-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}\right)^2-(\nabla\times\mathbf{A})^2-\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)^2\right\}+\rho\left(\mathbf{A}\cdot v-\phi\right). \tag{9.19}$$

Eq. (9.19) now defines the Lagrangian for a charged particle moving in a electromagnetic field produced by the charge ρ and current $\mathbf{j}$. The Lagrangian for the entire field, including the particles themselves, is given by integrating each individual charge with the volume using $\mathsf{L} = \int \mathsf{L} dV$ to give the field Lagrangian as,

$$\mathsf{L} = T + \int \left[\frac{1}{8\pi} \left\{ \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \right)^2 - (\nabla \times \mathbf{A})^2 - \left(\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} \right)^2 \right\} + \rho\,(\mathbf{A} \cdot v - \phi) \right], \quad (9.20)$$

which results in a description of both the equations of motion for charged particles embedded in the field and the equations for the electromagnetic field itself.

Eq. (9.20) represents the total Lagrangian of a set of particles interacting with the electromagnetic field. The dynamical variables of the particles form a discrete set involving the components of the position **r** and the velocity $d^2\mathbf{r}^2/dt^2$. For the electromagnetic field, it is the field potentials and the fields which represent the *generalized coordinates* of the Lagrangian.

The total Lagrangian has three terms, the Lagrangian for the particles, L_{obj}, the Lagrangian for the radiated field, L_{rad} and the Lagrangian for the interaction between the electromagnetic field and the particles, L_{int}.

The total Lagrangian is then given by,

$$\mathsf{L} = \mathsf{L}_{\text{obj}} + \mathsf{L}_{\text{rad}} + \mathsf{L}_{\text{int}} \quad (9.21)$$

where,

$$\mathsf{L}_{\text{obj}} = \sum_r \frac{1}{2} m_r \dot{q}_r^2 , \quad (9.22)$$

and

$$\mathsf{L}_{\text{rad}} = \frac{1}{2} \int \mathbf{E}^2 + \mathbf{B}^2 dv , \quad (9.23)$$

and

$$\mathsf{L}_{\text{int}} = \sum_r [\mathbf{j}_r \cdot \mathbf{A}_r - \rho_r \phi_r] . \quad (9.24)$$

Regrouping the radiation and interaction Lagrangian terms allows the re–introduction of the *Lagrangian Density*.

$$\mathsf{L} = \frac{1}{2}\left[\mathbf{E}^2 + \mathbf{B}^2\right] + [\mathbf{j} \cdot \mathbf{A} - \rho\phi], \quad (9.25)$$

and the following form for the *Standard Lagrangian*,

$$\mathsf{L}=\sum_r m_i\dot{q}_r^2+\int\mathsf{L}dv\ . \tag{9.26}$$

It should be noted that the interaction term is *local* with the current density at the point multiplied by the vector potential. In the radiated field Lagrangian the spatial derivatives of the potentials come about from **E** and **B**, which describes the coupling between the fields at each point. This coupling is the origin of the propagation of the free field.

§9.1. FIELD ENERGY DENSITY

Using the Hamiltonian expressions given in §8, Eq. (9.26) serves as the starting point for computing the Hamiltonian of the electromagnetic field in the absence of charged particles and field sources, that is the Hamiltonian of a freely propagating electromagnetic wave. The contribution of the first and last terms in Eq. (9.26) is determined by Eq. (8.44).

The remaining contribution will be from the terms in the field quantities alone,

$$\mathsf{H}\ =\int\mathsf{H}dV\ , \tag{9.27}$$

where,

$$\begin{aligned}\mathsf{H}&=\frac{1}{8\pi}\left\{2\mathbf{E}-2\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)\frac{\partial\phi}{\partial t}-\mathbf{E}^2+\mathbf{B}^2+\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)^2\right\},\\&=\frac{1}{8\pi}\left\{2\mathbf{E}^2+2\mathbf{E}\cdot\nabla\phi+\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)\left(\nabla\cdot\mathbf{A}-\frac{\partial\phi}{\partial t}\right)-\mathbf{E}^2+\mathbf{B}^2\right\},\\&=\frac{1}{8\pi}\left\{\mathbf{E}^2+\mathbf{B}^2+2\mathbf{E}\cdot\nabla\phi+\left((\nabla\cdot\mathbf{A})^2-\left[\frac{\partial\phi}{\partial t}\right]^2\right)\right\}.\end{aligned} \tag{9.28}$$

The total Hamiltonian given by Eq. (8.46) is now,

$$\begin{aligned}\mathsf{H}\ &=\mathsf{H}_S\left[q_r,\left(p_r-\int\sum_s\rho\frac{\partial v_s}{\partial\dot{q}_r}\mathbf{A}_s dV\right)\right]+\cdots\\&\cdots+\int\left[\rho\phi\frac{1}{8\pi}\left\{\mathbf{E}^2+\mathbf{B}^2+2\mathbf{E}\cdot\nabla\phi+(\nabla\cdot\mathbf{A})^2-\left(\frac{\partial\phi}{\partial t}\right)^2\right\}\right]dv.\end{aligned} \tag{9.29}$$

Where H_S is the Hamiltonian function of the dynamical system. When the field variables are taken as discrete, the ordinary Hamiltonian canonical equations can be used for the dynamical as well as for the electromagnetic variables. This first term of this Hamiltonian is the kinetic energy of the dynamically system. The sum of the last two terms under the integral sign is zero, due the Lorentz condition.

The first and fourth terms under the integral sign cancel since,

$$\begin{aligned}\int\left(\rho\phi+\frac{1}{4\pi}\mathbf{E}\cdot\nabla\phi\right)dv &= \int\left(\frac{1}{4\pi}\phi\,\nabla\cdot\mathbf{E}+\frac{1}{4\pi}\mathbf{E}\cdot\nabla\phi\right)dv,\\ &= \int\left\{\frac{1}{4\pi}\nabla\cdot\left(\phi\mathbf{E}\right)\right\}dv,\\ &= 0.\end{aligned} \tag{9.30}$$

If the field potential vanish sufficiently rapidly at infinity what remains is the radiation field energy density,

$$\int\frac{1}{8\pi}\left(\mathbf{E}^2+\mathbf{B}^2\right)dv\,. \tag{9.31}$$