Electric and magnetic fields effect matter on a range of scales from the atomic to the cosmic. Although these fields are only *visible* to humans in a narrow band of frequencies, they are responsible for the majority of chemical and biological effects that make up life as we know it. Electric fields are produced by electric charge and magnetic fields are produced by motion of electric charge. Electric fields can also be produced by a changing magnetic field and magnetic fields can be produced by a changing electric field. These two fields are coupled into one field, the electromagnetic field described by Maxwell's equations.

Maxwell's codification of the field equations transcends their physical description. Einstein put this achievement in perspective in the commemoration of the 100th birthday of James Clerk Maxwell:

> We may say that, before Maxwell, physical reality in so far as it was to represent the process of nature, was thought of as consisting in material particles, whose variations consist only in movements governed by partial differential equations. Since Maxwell's time, physical reality has been thought of as represented by continuous fields, governed by partial differential equations, and not capable of any mechanical interpretation. This change in the conception of reality is the most profound and most fruitful that physics has experienced since the time of Newton… [Eins31]

Since the time of Rene Descartes (1596–1642), natural philosophers have speculated how the electric force is transmitted through space [Whit51], [Cott92]. [1] Karl Friedrich Gauss (1777–1855) wrote Wilhelm

[1] René Descartes' is one of several 17th century *natural philosophers* who considered the question of *action–at–a–distance* including Galileo, Newton and Huygens. Although many of Descartes' explanations did not survive, he laid the foundation for later theories that did. In practice Descartes' method was a mixture of rationalism and empiricism, but his theoretical approach tended to be entirely rationalist when he claimed that when the first principles or *simple natures* of philosophy are clearly and distinctly perceived, all other truths are deduced from them [Hess61].

Since Descartes saw verification of the *simple natures* by comparing them with experience as a *mere formality*, there was no process by which his theories could be tested. He did use observation and experimentation to arrive at his *simple natures* through the deductive process put forward by Aristotle and Newton. He stated that his chief innovation to physics, when compared to Aristotelian science was the ...

Weber (1804–90) [2] in 1845 to remark that...

> ...he had proposed to himself to supplement the known forces which act between electrical charges by other forces, such as would cause electric actions to be propagated between charges at finite velocity.

In a paper by Gauss published posthumously in 1867, the concept of electric action propagating through space is presented. Maxwell stated that...

> ...this paper first introduced into mathematical science that idea of electric action carried by means of a continuous medium... [Maxw65], [Buch85].

The concept of a field was introduced by Faraday [Fara65], [Agas71] and extensively developed by Maxwell [Maxw65], [Ever74], [Larm37], [Agas68]. [3] In Maxwell's theory electric charge is regarded as the source

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> ...explanatory principles .. that he clearly and distinctly perceived. [Hess61].

Descartes' scientific method was laid out in 1637 in his *Discours de la Methode Pour bien conduire fa raison & cherches la veritè dans kes* sciences (*Discourse on the Method of Properly Guiding the Reason in the Search of Truth in Science*). His method consists of:

(a) accepting only what is clear in one's own mind as to exclude any doubt,

(b) splitting large difficulties into smaller ones,

(c) arguing from the simple to the complex,

(d) checking when one is done [Davi86a].

Appended to *Discours* are three essays on which Descartes gives examples of discoveries may by the use of his *methods*. The first appendix title *Optics* presents the laws of the refraction light, which had earlier been discovered by Willebrord Snell. The second appendix discusses meteorology. The third appendix presents analytical geometry.

[2] In 1855 Weber and Kohlrausch determined the limiting velocity in Weber's electromagnetic theory to be $c = 439{,}450 km/s$ [Webe56], [Webe93], [Doug90], [Jung86].

[3] James Clerk Maxwell was the son of a Scottish nobleman who encouraged Clerk's curiosity by taking him to see the manufacturing plants being developed in the 1830's and 1840's. During *high school* Maxwell became interested in mathematics and at age 15 published a paper in the *Proceedings of the Royal Society of Edinburgh* titled "On the Description of Oval Curves." [Macd64]. In 1857 he wrote to Faraday commenting on some of his ideas and Faraday replied with encouragement. In 1873 Maxwell published his treatise *Electricity and Magnetism* [Maxw65] in which he unified electricity and magnetism through four equations, predicted electromagnetic wave propagation, calculated the velocity of light and conjectured that light was an electromagnetic wave phenomena [Ever74].

Maxwell's equations were experimentally confirmed by Heinrich Hertz in two papers

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of the field and the electric and magnetic fields are fields of *force*, represented as vector fields. Laboratory experiments have shown that electric currents produce magnetic fields and vice versa. Electricity and magnetism are not distinct forces but are in fact a single electromagnetic force. The first steps to unify the theory of *electromagnetism* were taken in the late 1800's [Buch85]. Maxwell set out to write down a set of equations, which when taken with the strengths of the electric and magnetic fields and the strengths and arrangements of the sources of these fields, would result in the equations of motion for a test charge placed in the field.

Maxwell succeeded in showing that the force a charge exerts on another charge and the energies of each charge could be expressed not only in terms of the magnitudes of the charges and their positions in space, but in terms of a *stress energy tensor* that was defined throughout the medium, even if that medium was free space. The need for a medium to convey the electromagnetic field was fundamental to Maxwell's concept of *waves*. Like velocity waves in hydrodynamics or the stress and strain field in elasticity, these fields were not considered to exist by themselves but were somehow considered to be vibrations of an underlying *luminiferous aether*, whose properties were akin to a elastic solid [Larm37], [Whit60], [Buch85]. The electromagnetic waves were secondary. The aether could exist without the waves, but the waves could not exist without the aether. [4]

Years after the discovery by Maxwell of the *wave equations*, an experiment in 1886 by Albert Abraham Michelson (1852–1931) and Edward Williams Morely (1838–1923), showed that the aether did not exist. [5] As a result, the electromagnetic field can be viewed as an entity of

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published in 1881, "On Electromagnetic Waves in Air" and "On Electric Radiation."

[4] The spelling *aether* with the initial diphthong is an arcane reversion that even Michelson and Maxwell did not use. However modern texts on the history of electromagnetism [Whit60], [Swen72] make use of this spelling. The word aether is related to the Greek αιθηρ which means upper air or sky and relates to the refined fire of the empyrean [Shan64].

[5] The existence of the *aether* was accepted by many scientist of the time as a *logical* conclusion of Maxwell's theories, since some form of media would be necessary to *carry* the electromagnetic waves. One consequence of the aether would be that the earth's motion through a motionless aether produce a drag effect resulting in measurable differences in the speed of light depending on the direction it is traveling relative to the earth. This idea was put to a stringent test by Michelson and Morely in 1881 [Mich81], [Livi88]. They used the earth as a moving reference frame and compared the round trip speed of light along the line of the earth's motion with the speed of light perpendicular to the line of the earth's motion. They found no evidence for such a drag effect and no evidence for relative motion

its own, as real as physical matter. The *essence* of electromagnetic field theory is that the field *does* have properties that usually as associated with matter. The four equations, in *Rationalized* units, describing the electromagnetic theory developed by Maxwell are given below. [6]

§3.1. MAXWELL'S 1ST EQUATION — COULOMB'S LAW

The first Maxwell equation deals with static electric fields. Given a collection of charges, surrounded by closed surface, the number of *lines of force* passing through a unit area, A, on the surface is given by,

$$N = \int_A \mathbf{E} \cdot d\mathbf{A} = \sum_{i=1}^{n} q_i \ . \qquad (3.1)$$

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between the aether and the earth [Lore95]. This was one of the most important *negative* experiments performed and resulted in the 1907 Nobel prize being awarded to Michelson [Bagg92]. Hendrich Antoon Lorentz (1853–1928) proposed an explanation to this negative experiment by introducing the Lorentz *contraction* of the light as it travels parallel to the earth's motion, which *canceled* of the expected change in the interference patterns proposed by Michelson and Morley [Swen88]. Although Lorentz's explanation of the null experiment proved to be incorrect, portions of his concept were later included in Einstein's 1905 theory of relativity. (This connection between Lorentz, Michelson and Einstein is somewhat controversial [Buch88], [Swen88].) Einstein showed that both Michelson's and Lorentz's concepts could be reconciled by using newer ideas. Several *modern* experiments have been performed to confirm the negative of Michelson–Morley's observations [Pano66], [Jase64].

[6] The *rationalized* units is used in this monograph as one of the five systems of units found in electromagnetic field theory. The rationalized units notation provides the simplest form in which to *describe* the workings of electromagnetic field theory, since no actual calculations are necessary in the approach taken here [Brid31]. The desirable features of a system of units in any field of study are clarity and convenience. By choosing the rationalized units clarity is provided by the absence of terms dealing with permeability and permitivity. One additional *simplification* has also taken place in this paper, which makes the notation convenient, but erroneously describes the electromagnetic field in a practical sense. The notation for the magnetic field used here is $\mathbf{B}$, when in fact it is $\mathbf{H}$, where $\mathbf{H} = 1/\mu_0 \, \mathbf{B} - \mathbf{M}$, where $\mathbf{M}$ is the magnetization term and μ_0 is the field permitivity. The electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ are simplified to $\mathbf{E}$. This results in a technically incorrect formulation of Maxwell's equations for the propagation of electromagnetic waves in an isotropic medium (air), but provides a simplified notation for the development of the quantum mechanical formulation found later in the monograph. This form of Maxwell's equations, in which the properties of matter are not considered, is properly called the *microscopic* Maxwell equations in contrast to the original *macroscopic* equations.

The final *adjustment* to the notation is to set $\hbar = c = 1$ resulting in *natural units*. In the MKS system of units this allows the constants μ_0 and ε_0 to be assigned values of 1. The factor 4π can then be moved from the field equations to the force equations.

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The expression in Eq. (3.1) is referred to as *Gauss's Law*. Charge is always given in discrete units of e, but it is reasonable to assume when dealing with macroscopic phenomena that charge can be described by a continuous distribution function, the charge density, $\rho(\mathbf{r})$. Gauss's Law can then be rewritten as,

$$\int_A \mathbf{E}\cdot d\mathbf{A} = \int_V \rho dV \ . \tag{3.2}$$

The left side of Gauss's Law can be rewritten again using the divergence theorem of vector calculus which states,

$$\nabla\cdot\mathbf{E} = \lim_{V\to 0}\left(\int_A \mathbf{E}\cdot dA\right)\frac{1}{V}, \tag{3.3}$$

which is called Gauss's Theorem (different from Gauss's Law). The divergence theorem (Gauss's Theorem) is used to express a surface integral as a volume integral of the divergence of the electric field $\nabla\cdot\mathbf{E}$, and is valid for any volume V. The *divergence* of a vector field measures the net amount of charge entering or leaving a small volume. For the electric field, the divergence is a scalar value representing the number of lines of force passing through the surface surrounding a charge. If the surrounding surface does not contain any charge, then the number of lines entering the volume is equal to the number of lines leaving the volume resulting in $\int_A \mathbf{E}\cdot dA = 0$. If the surrounding surface contains charge, then Gauss's Theorem produces a nonzero result.

Equating the expression for the number of lines passing through the surface of the total charge within the surface gives,

$$\int_V \nabla\cdot\mathbf{E}dV = \int_V \rho dV \ . \tag{3.4}$$

Differentiating both sides with respect to the volume surrounding the charge leads to Maxwell's first equation or Coulomb's Law,

(I) $$\nabla\cdot\mathbf{E} = \rho \tag{3.5}$$

which states that,

Flux of $\mathbf{E}$ *through a closed surface = charge inside the closed surface.* [7]

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[7] Experiment shows that in the simultaneous action of several charges, $e_1, e_2, e_3 \ldots$ on a test charge their field contributions, as well as their force actions, behave according to

All of electrostatics stems from the quantitative statement of Coulomb's Law. Coulomb showed experimentally that the total force produced on a small charged body by a number of other small charged bodies placed around it was the vector sum of the individual two–body forces. [8]

§3.2. MAXWELL'S 2ND EQUATION — ABSENCE OF MAGNETIC MONOPOLES

The second Maxwell equation deals with static magnetic fields. The magnetic *intensity* $\mathbf{B}$ corresponds to the electrical intensity $\mathbf{E}$ even though $\mathbf{B}$ is usually called the magnetic *induction*. Laboratory experiments in the late 19th century showed that the properties attributed to electrical charges can also be applied to magnetic *poles*. The number of lines of magnetic force passing through a surface can be written as,

$$N = \int_A \mathbf{B} \cdot dA = \int_V \nabla \cdot \mathbf{B} dV \ . \tag{3.6}$$

If a surface is placed around a distribution of *magnets*, with each magnetic pole composed of a *north* and *south* pole, the number of lines of force going outward through the surface from the north poles is equal to the number of lines of force coming inwards to the south poles, such that,

$$N = \int_V \nabla \cdot \mathbf{B} dV = 0 , \tag{3.7}$$

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the vector law of addition; $\oint E_n ds = 4p \sum_{e_j \in V} e_j$, where the summation is applied to the charge inside the volume V. Given a large number of charges, within the volume element, using the ratio of volume, dv, to a total charge ρdv, and applying the electric flux theorem $\oint u_n ds = \nabla \cdot u(v_i)$, gives the differential form of the electric flux theorem as $\nabla \cdot \mathbf{E} = \rho$ which is Coulomb's Law, Poisson's generalization of the inverse square law and Maxwell's first equation as well as Gauss' theorem.

[8] The concept of *flux* was first used by Maxwell in his analogy between the flow of heat and the flow of electricity. Maxwell *borrowed* this idea from William Thomson (1824–1907), later Lord Kelvin. Thomson had shown that the equations describing static electricity are of the same form as those describing the flow of heat [Larm37]. This analogy of electrical flux with heat flux can be extended to the concept of a *streamline*. Maxwell suggested that Faraday's lines of force were similar to *streamlines* in the flow pattern of a fluid.

Thompson was considered the most eminent experimental physicist in Britain. Thompson was selected to head the Cavendish (Henry Cavendish (1731–1810)) laboratory at Cambridge University. However he wished to in his native Glasgow and instead the Cavendish Professorship went to another Scotsman James Clerk Maxwell, who at age 39 was living in retirement on his estate at Glenair [Wein90].

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which gives Maxwell's second equation as,

$$\text{(II)} \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{3.8}$$

which states that the,

Magnetic flux **B** *through a closed surface = zero,*

which implies the absence of magnetic monopoles, which in turn implies that the lines of magnetic flux produced by the electric field do not start or stop.

§3.3. AMPÈRE'S LAW FOR STEADY STATE FIELDS

Maxwell's third and fourth equations involve the description of time varying electric and magnetic fields. If a continuous distribution of charge is placed in motion, resulting in the production of current, the current density can be described in a manner similar to the electric and magnetic *intensity* of the previous two equations.

The current density **j** per unit area *A* can be used to define a *current*, $i = \int_A \mathbf{j} \cdot dA$, which can be rewritten as,

$$\int_S \mathbf{B} \cdot dS = \int_A \mathbf{j} \cdot dA\,. \tag{3.9}$$

The *curl* of a vector field is defined as,

$$\nabla \times \mathbf{B} = \lim_{A \to 0} \left(\int_S \mathbf{B} \cdot dS \right) \frac{1}{A}\,, \tag{3.10}$$

where *dS* is an element of the path surrounding the unit area *A*. If follows that,

$$\int_S \mathbf{B} \cdot dS = \int_A \nabla \times \mathbf{B} \cdot dA\,. \tag{3.11}$$

This result is called *Stokes Theorem* and can be rewritten as,

$$\int_A \nabla \times \mathbf{B} \cdot dA = \int_A \mathbf{j} \cdot dA\,. \tag{3.12}$$

Differentiating both sides of Eq. (3.12) gives,

$$\nabla \times \mathbf{B} = \mathbf{j}\,, \tag{3.13}$$

which is known as Ampère's Law and restates that *currents* are *generated* by the motion of charge.

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The problem with Eq. (3.13) is that although the magnetic field and the current is generates are *linked* by Ampère's Law, it contains no reference to how the current varies with time, since Ampère's Law was derived from *static* fields.

If a closed surface is place around an electrical circuit, the net current flowing through the surface is defined as, $\int_A \mathbf{j}\cdot dA$. This integral will have a positive value when the net flow of current of outward through the surface. This flow of current then must be balanced by an equivalent reduction of the electrical charge within the enclosing volume. The total charge within the volume is defined as $\int_V \rho \cdot dV$ and the rate of loss of the this over time is $(\partial/\partial t)\int_V \rho \cdot dV$.

The loss of this charge produces a current defined as,

$$\int_A \mathbf{j}\cdot dA = -\frac{\partial}{\partial t}\int_V \rho\, dV \quad (3.14)$$

Using the definition for the *divergence* of a vector field gives,

$$\int_V (\nabla\cdot\mathbf{j})\, dV = \int_A \mathbf{j}\cdot dA, \quad (3.15)$$

which when substituted into Eq. (3.14) gives,

$$\int_V (\nabla\cdot\mathbf{j})\, dV = -\frac{\partial}{\partial t}\int_V \rho dV \ . \quad (3.16)$$

Differentiating both sides of Eq. (3.16) gives,

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t}, \quad (3.17)$$

which is called the *continuity equations* and will be discussed in detail later.

Returning to Ampère's Law and taking the divergence of both sides gives,

$$\nabla\cdot(\nabla\times\mathbf{B}) = \nabla\cdot\mathbf{j}. \quad (3.18)$$

The left hand side of this equation is of the form $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})$ where **A** and **B** are vectors. The term $(\mathbf{A}\times\mathbf{B})$ is a vector a right angles to both **A** and **B**. The scalar product of **A** with $(\mathbf{A}\times\mathbf{B})$ must be zero, since the two vectors are at right angles, resulting in $\nabla\cdot(\nabla\times\mathbf{B}) = 0$. This corresponds to

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the physical situation of the curl of **B** follows the *lines of magnetic force* for which the divergence is zero.

The result of this development is that Ampère's Law implies $\nabla \cdot \mathbf{j} = 0$. Using Eq. (3.17), Ampere's equation then implies $\partial\rho / \partial t = 0$. This results in the confirmation that Ampère's Law holds for steady currents where the rate of flow of charge is constant.

§3.4. MAXWELL'S 3RD EQUATION — AMPERE'S LAW

In Maxwell's time the magnetostatic equation, $\nabla \times \mathbf{B} = \mathbf{j}$ (Eq. (3.13)) was *extended* to include the concept of a *displacement current*. In modern physics, this extension would be considered a postulate. With the advantage of historical hindsight the divergence of Eq. (3.15) is zero — since the divergence of any curl is zero.

When the divergence of a vector filed is zero, it means that the *lines of force* for that field close in on themselves. In the case of the magnetic field, current flows is closed loops when static fields are present, but not with time varying fields. Maxwell understood this process through the concept of Faraday's *polarization current*. Faraday discovered this phenomenon of polarization — the separation of charges in a dielectric placed in an electric field. Since charges in motion are defined as a current, the process of polarization implies the existence of a *polarization current*. Maxwell developed these ideas in a series of papers during 1861–1862 [Whit51], [Maxw62].

In Maxwell's paper [Maxw62] the concept of an elastic media played an important part in t he development of the displacement current paradigm. Maxwell describes how energy is stored in insulators, by *displacing* the electric particles from their equilibrium position by the action of the electric field. The movement of these electric charges produces a current. Since the media was assumed to be elastic, it was possible to calculate the velocity of the electric charges as they traveled through the media. The displacement current would then be proportional to the electric field strength. This calculation involved two constants α and β, which were unknown to Maxwell. These constants were later associated with the permitivity and permeability of free space.

Using the steady state condition of Ampere's Law $\nabla \times \mathbf{B} = \mathbf{j}$, this equation can be altered for electromagnetic fields that vary with time. Taking the divergence of both sides of Ampere's Law, $\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot \mathbf{j}$ gives $\nabla \cdot \mathbf{j} = 0$, since the divergence of the curl of a vector fields vanishes identically. It can be shown that in general, $\nabla\cdot\mathbf{j}=-\partial\rho/\partial t$ (the continuity equations developed in §3.8) the term $\nabla\cdot\mathbf{j}$ vanishes *only* in the case that the charge density is static. Therefore, Ampere's Law given as $\nabla\times\mathbf{B}=\mathbf{j}$ is insufficient for time dependent – charge density varying – fields. Maxwell made several attempts to modify Ampere's Law. In 1861 Maxwell noted his ideas in a letter to Sir William Thomson (Lord Kelvin), but did not develop the idea fully until 1865. Maxwell made the substitution $\mathbf{j}\to\mathbf{j}+\partial\mathbf{E}/\partial t$ so that Ampere's Law becomes $\nabla\times\mathbf{B}=\mathbf{j}+\partial\mathbf{E}/\partial t$. The continuity law can be recovered from this equation by taking the divergence of both sides, so that $\nabla\cdot(\nabla\times\mathbf{B})=\nabla\cdot\mathbf{j}+\nabla\cdot(\partial\mathbf{E}/\partial t)$ to give $0=\nabla\cdot\mathbf{j}+\nabla\cdot(\partial\mathbf{E}/\partial t)$. By interchanging the space and time derivatives of $\mathbf{E}$, gives, $\nabla\cdot\mathbf{j}+\partial/\partial t(\nabla\cdot\mathbf{E})=0$. Using Coulomb's Law, $\nabla\cdot\mathbf{E}=\rho$, the continuity law can be restored as $\nabla\cdot\mathbf{j}+\partial\rho/\partial t=0$. This provides for the conservation of charge where the original Ampere's Law did not [Nive66].

Since polarization currents exist in ordinary matter — capacitors and dielectrics — Maxwell assumed that similar current could exist in the *ether*. In order to salvage Maxwell's original magnetostatic equation a displacement current was introduced, such that the magnetostatic equation is given as,

$$\nabla\times\mathbf{B}=\mathbf{j}+\mathbf{j}_D, \tag{3.19}$$

where $\mathbf{j}_D$ is the displacement current, which implies,

$$\nabla\cdot\mathbf{j}_D=-\nabla\cdot\mathbf{j}. \tag{3.20}$$

Since $\nabla\cdot\mathbf{j}=-\partial\rho/\partial t$, it follows $\nabla\cdot\mathbf{j}_D=-\partial\rho/\partial t$. Using Maxwell's first equation, $\nabla\cdot\mathbf{E}=\rho$ and differentiating both sides gives,

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E})=\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}=\frac{\partial\rho}{\partial t}=\nabla\cdot\mathbf{j}_D, \tag{3.21}$$

which can be rewritten as,

$$\nabla\cdot\left(\frac{\partial\mathbf{E}}{\partial t}-\mathbf{j}_D\right)=0, \tag{3.22}$$

which can be reduced to,

$$\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j}_D. \tag{3.23}$$

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Substituting this result into Eq. (3.19) gives Maxwell's equation as,

$$\text{(III)} \qquad \nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial\mathbf{E}}{\partial t}, \qquad (3.24)$$

which states the,

(*Line integral of* $\mathbf{B}$ *around a loop*) = *Current thorough the loop*

$+\dfrac{\partial}{\partial t}$ (*the flux* $\mathbf{E}$ *through the loo*p);

which is Ampère's Law (1775–1836) adjusted for time varying fields. [9] Maxwell was inspired by the equivalence of the displacement current and ordinary currents in order to generalize Ampère's Law. The addition of this displacement current allowed Maxwell to predict the existence of electromagnetic waves [Eyge72].

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[9] The four equations given above are referred to as Maxwell's *Free Space* equations, but it must be remembered that these equations are actually *assumptions* of the Maxwell theory. Equations I, II, and III have been derived from experiments performed in steady state situations where fields and currents are not changing with time. Maxwell was largely responsible for putting these laws in the form of differential equations and contemplated the possibility that they were all valid even in time-dependent situations and realized that Ampère's Law (III) was inconsistent with the continuity equation (Eq.(19)). Maxwell saw that if Ampère's Law is modified by the addition of a time derivative (which leaves it unaltered in a steady state) to become $\nabla\times\mathbf{B} - \partial\mathbf{E}/\partial t = \mathbf{j}$, then there is a set of equations which are mathematically consistent even for $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ varying with time. In Maxwell's time the first magnetostatic equation (III) was given by $\nabla\times\mathbf{B} = \mathbf{j}$. Taking the divergence of this equation, since the divergence of the curl is always zero, gives $\nabla\cdot\mathbf{j} = 0$. The fact that the divergence of a vector field is zero means that the field lines close on themselves. Current does flow in closed loops when static fields are present, but not when the fields vary with time. However, Maxwell's conception of this process somewhat different then modern formulations. Faraday had discovered and Maxwell knew about the phenomenon of *polarization currents*, or the separation of charges in a dielectric when placed in an electric field. Charges must move to separate, and since charges in motion constitute a current, this motion was referred to as the *polarization current*. Since polarization currents were know to exist in ordinary matter, it was natural for Maxwell to assume that similar currents could exist in the ether (in Maxwell's terms) that filled empty space [Whit60], [Buch85]. Maxwell then *salvaged* what appears to have been considered the basic equation for current flow in loops. From $\nabla\cdot\mathbf{E} = \rho$, the time rate of change on the charge density is given by $\partial\rho/\partial t = \partial(\nabla\cdot\mathbf{E})/\partial t$. Placing this result into the continuity equation $\nabla\cdot\mathbf{j} + \partial\rho/\partial t = 0$ gives $\nabla\cdot(\mathbf{j} + \partial\mathbf{E}/\partial t) = 0$. The second term, $\partial\mathbf{E}/\partial t$, has the proper dimensions to be considered a current density and is called *Maxwell's Displacement Current*. The physical interpretation of this additional term is the description of the displacement current which produces a circulation in the magnetic field [Jack75].

§3.5. MAXWELL'S 4TH EQUATION — FARADAY'S LAW OF INDUCTION

Michael Faraday found that when he magnetic *flux* passing through a conducting circuit was changed, a current was observed to flow in the circuit. The current was *induced* in such a way that it opposed the original flow of current in the circuit. This behavior is called Lenz's Law and states that *the change of flux through a loop induces an electromotive force and associated current, which opposes the original flux*. The induced current is a secondary effect whose value depends on the resistance of the circuit. This change in the magnetic flux results in an electromotive force $\mathbf{E}$, which produces the observed current,

$$\mathbf{E} = -\frac{dN}{dt}, \tag{3.25}$$

where N is the density of the magnetic lines of force.

The left hand side of Eq. (3.25) can now be converted into a form compatible with Maxwell's equations by defining the electromotive force as the force acting on a collection of electric charges,

$$\mathbf{E} = \int_S \mathbf{E} \cdot dS. \tag{3.26}$$

Using the previous definition for the curl of a vector, the expression for the electromotive force can be rewritten as,

$$\int_S \mathbf{E} \cdot dS = \int_A \nabla \times \mathbf{E} \cdot dA, \tag{3.27}$$

where dA is the unit are enclosed by the circuit. The right hand side of Eq. (3.27) can be rewritten by inserting the definition of the number of lines of magnetic force, $N = \int_A \nabla \times \mathbf{B} \cdot dA$ and differentiating both sides to give,

$$\frac{dN}{dt} = \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot dA. \tag{3.28}$$

Combing both sides of the rewritten Eq. (3.27) gives,

$$\int_A \nabla \times \mathbf{E} \cdot dA = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot dA. \tag{3.29}$$

Differentiating both sides gives, the fourth Maxwell equation is,

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$$\text{(IV)} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad (3.30)$$

which states the,

$$(\textit{Line integral of } \mathbf{E} \textit{ around a loop}) + \frac{\partial}{\partial t}(\textit{the flux } \mathbf{B} \textit{ through the loop}) = 0;$$

which is Faraday's Law of induction.[10] In 1831 Faraday made the first quantitative observations of time–dependent electric and magnetic fields. He observed that a transient current is induced in a circuit if the steady current flowing in an adjacent circuit is turned off or on; or the adjacent circuit with a steady current flowing is moved relative to the first circuit; or a permanent magnet is moved into or out of the circuit. faraday interpreted the transient current flow as being caused by a changing magnetic flux linked by the two circuits. The changing flux induces an electric field around the circuit, the integral of which is the *electromotive force*. According to Ohm's Law, this electromotive force causes a current to flow in a circuit [Agas71].[11]

Faraday's Law is a kind of *relativity* effect with respect to the motion of magnetic fields [Eins09a]. If the circuit in a magnetic field is moved with velocity v, the charges surround this circuit will be acted upon by a force $v \times \mathbf{B}$. If the circuit is held stationary and the *magnets* are moved, the same effect should occur. But since the charges are at rest, they must be acted on by the electric force $q\mathbf{E}$ generated by the circuit. The equality of these two motions is expressed in Faraday's Law integral form,

10 Equations (III) and (IV) are commonly referred to as the *Curl* equations, leaving Equations (I) and (II) to be referred to as the *Divergence* equations.

11 Faraday found experimentally that a nonconservative electric field accompanies varying magnetic fields. Contrary to the description in many texts, Faraday's law of induction is not the consequence of the law of conservation of energy applied to the overall energy balance of current's in magnetic fields. Given a circuit of resistance $\mathbf{R}$ carrying current $\boldsymbol{j}$ resulting in a electromotive force E, the magnetic flux Φ_m surrounding this circuit is $\Phi_m = \int \mathbf{B} \cdot dS$, where the surface of integration is bounded by the circuit. When the current changes with time, there are experimental observations that show $j\mathbf{R} - E = -\partial \Phi_m / \partial t$. This means that the current in the circuit is different than the current *predicted* by Ohm's law, by an amount equal to the negative time rate of change of the magnetic flux through the circuit. This expression is an experimentally derived law, not a law deduced from any theoretical understanding available to Faraday [Pans55].

$\frac{d}{dt}\int_F \mathbf{B}\cdot df = -\oint_C \mathbf{E}\cdot ds$ and Ampère's Law $\int_F \mathbf{C}\cdot df = \oint_C \mathbf{B}\cdot ds$, where $\mathbf{C} = \mathbf{j} + d\mathbf{B}/dt$. For a closed surface Faraday's Law and Ampère's Law become $\oint \mathbf{B}\cdot df$ = constant in time and $\oint \mathbf{C}\cdot df = 0$. The second of these equations indicates that the electric current lines are closed. Because **B** is finite and the surrounding space is homogeneous, the same must hold for magnetic lines in agreement with $\oint \mathbf{B}\cdot df = 0$. This relation will be formalized in a following section, by developing the concept of *continuity*.

§3.6. NEWTON–LORENTZ FORCE EQUATION

The description of the interaction between charged particles by the intermediary electromagnetic field originated with Faraday and Maxwell. The field is produced by the charged particles — whose existence is *assumed* — and is measured by the acceleration it produces when acting on other charged particles. [12]

One approach to the development of this *force equation* is through the experimental observation that when a wire carrying a current, i, is place in a magnetic field a force $d\mathbf{F}$ is exerted on a short length of wire $d\mathbf{s}$, such that,

$$d\mathbf{F} = i\left(d\mathbf{s}\times\mathbf{B}\right). \tag{3.31}$$

This force is a *right angles* to the current flowing in the wire *and* is at right angles to the magnetic field **B**, since the force is the result of a vector product.

In the early nineteenth century experiments were performed to observe what happens when a current flows through a conductor. [13] Two

[12] The interaction of the electromagnetic field generated by a charged particle on *the charged particle itself* is one of the intractable problems of electrodynamics. An accelerating particle produces a field itself which changes the external field in which the particle is moving, which in turn will effect the motion of the particle. The theory of interacting particles and the fields they generate is *intricate* and involved, both conceptually and mathematically. The solution to this *problem* is provided by the relativistic formulation of Maxwell's equations and Quantum Electrodynamics.

[13] The law describing the behavior of the magnetic field produced by the current is referred to as the Biot–Savart Law, but the assignment of this name is still open to question. Following Ørsted's announcement of the effects of a current on a permanent magnet in 1820, Ampère announced the observation of similar forces produced by a current or other currents flowing in a nearby wire. Biot and Savart presented the first quantitative

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French scientists, Jean–Baptiste Biot (1774–1862) and Félix Savart (1791–1841) found a relationship between the current flowing in a wire and the magnetic field produced by the flowing current. [14] The magnetic field $\mathbf{B}$ due a current i flowing along a segment $d\mathbf{s}$ is given by,

$$d\mathbf{B} = \frac{\mu_0 i}{4\pi r^3}\left(d\mathbf{s}\times\mathbf{r}\right). \tag{3.32}$$

This equation is known as the Biot–Savart law and describes the *inverse square* relationship between current and magnetic field intensity, with the magnetic field vector *pointing* at right angles to the current flow. Both the Biot–Savart equation and the equation for the force due to a flowing current involve the flow of current through a segment of wire, $id\mathbf{s}$. Since this current is the result of the movement of charge q along with segment of wire, the current is now $i = dq/dt$. If the charge is moving with velocity v then $d\mathbf{s} = vdt$, which results in $id\mathbf{s} = \left(dq/dt\right)vdt = dqv$.

The Biot–Savart can be rewritten is the form,

$$d\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left(dqv\times\mathbf{r}\right), \tag{3.33}$$

and integrating over the charge volume,

$$\mathbf{B} = \frac{\mu_0 q}{4\pi r^3}\left(v\times\mathbf{r}\right). \tag{3.34}$$

The magnetic force equation, Eq. (3.31) can be rewritten in a similar manner to give,

$$d\mathbf{F} = dq\left(v\times\mathbf{B}\right), \tag{3.35}$$

and after integrating gives,

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statement for the special case of a current flowing in a straight wire. Ampère later formulated a general description of this effect for currents flowing in arbitrary paths [Whit60], [Mott22].

[14] In 1816 Biot published a work on the experimental methods in physics *Traité de Physique expérimental et mathématique* (*Treatise on experimental and mathematical physics*). This work established the importance of precise experimentation, stressing the need to improve the accuracy and precision of measurements by introducing new procedures and instruments to laboratory work. Biot described a mathematical and experimental method in which quantification of the results became the paradigm of science [Harm82].

$$\mathbf{F} = q\left(v \times \mathbf{B}\right). \tag{3.36}$$

Inserting Eq. (3.34) into Eq. (3.36) gives an expression for the magnetic force as,

$$\mathbf{F}_{magentic} = \frac{\mu_0 q^2}{4\pi r^3}\left[v \times \left(v \times \mathbf{B}\right)\right]. \tag{3.37}$$

Comparing this to the electric force ,

$$\mathbf{F}_{electric} = \frac{q^2}{4\pi\varepsilon_0 r^3}\mathbf{r}, \tag{3.38}$$

allows the magnitudes of the two forces (ignoring the vector nature of the electric and magnetic quantities) to be *compared*,

$$\frac{\mathbf{F}_{magnetic}}{\mathbf{F}_{electric}} = \varepsilon_0\mu_0 v^2 = \frac{v^2}{c^2}, \tag{3.39}$$

where $1/\varepsilon_0\mu_0 = c$ has the dimensions of a velocity and its magnitude is the speed of light is a vacuum.

Using the previous definition of the electric force, $\mathbf{F}_{electric} = q\mathbf{E}$ and adding this to the magnetic force gives, an expression for the force applied to a charged particle in the presence of the electric and magnetic fields can be written as the Newton–Lorentz equation. Experimentally it is determined that a particle carry charge *q*, and moving in a vacuum with a velocity *v*, experiences a force **F** which is given by,

(V) $$\mathbf{F}_{total} = \mathbf{F}_{electric} + \mathbf{F}_{magnetic} = q(\mathbf{E} + v \times \mathbf{B}). \tag{3.40}$$

Although this force equation was derived for currents flowing in a wire, it is also valid of charged particles moving in electric and magnetic fields in free space. [15]

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[15] The *discovery* of the Lorentz force $F = e\left(\mathbf{E} + v \times \mathbf{B}\right)$ was made in a 1895 paper [Lore85] written by Lorentz, which lays the foundation for the Special Theory of Relativity. It was in this paper that Lorentz proved the concept of *corresponding events* [Pais86] in which the transformation of spatial and temporal coordinates between moving reference frames was first described mathematically. The Lorentz transformations were given in 1899 to a factor of ε. In 1904 this factor was fixed to unity (1), but Lorentz made an error in transforming the velocity components in the inhomogeneous Maxwell equations. Because of this Lorentz did not obtain a covariant solution to Maxwell's equation which was needed to move forward with special relativity.

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§3.7. COUPLING STRENGTH OF THE ELECTROMAGNETIC FIELD

The electromagnetic field results in the discovery that fields and forces are tightly connected. It appears that the mass of particles is directly connected from the energies of the fields from which the particles serve as the source — the mass of the electron appears to be largely the mass corresponding to the energy of the electromagnetic field generated by its charge. From this view, a particle can be seen as a source or singular point in the field. The field of force can then be considered to consist of particles, such as the electromagnetic field can be described as a spectrum of photons emitted and absorbed by charged sources [Adai87].

In the previous section an expression of the force *felt* by a charged particle traveling in an electromagnetic field was given by Eq. (3.40). The relative strength of the electromagnetic force, weak force, strong force and gravitational force becomes a *dimensional analysis* problem — what units can be used to define *force* when the forces involved have very different characteristics. [16] In many texts the electromagnetic *coupling constant* is simply stated as a fact, without derivation nor a description of its importance to the understanding of the electromagnetic force. This section provides a *diversion* in order to describe the dimensional analysis approach to coupling constants and their importance to the understanding of all forces of nature.

There are several methods for approaching the electromagnetic field coupling strength problem. One approach involves the development of the axiomatic description of the underlying quantum field theory. Such a description is provided in [Feyn62]. In Feynman's description the properties of the electromagnetic field are extended are made consistent with the tenants of quantum mechanics. Although these techniques have produced accurate results, there are still many problems to be solved. In this section a dimensional analysis approach will be used to develop the coupling constants required for a consistent description of the forces of

[16] The foundations of dimensional analysis were laid out in Fourier's *Theorie Analytique de la Chaleus*, published in Paris in 1822. This work contains the formulation of Fourier Analysis as well. Fourier realized that every physical quantity...

> ... has one dimension proper to itself, and that the terms of one and the same exponent of dimension. We have introduced this consideration ... to verify the analysis ... it is the equivalent of the fundamental lemmas which the Greeks have left us without proof [West88].

nature. [17]

For an electron traveling in an electromagnetic field its *coupling strength* can be compared to the corresponding centripetal force of an orbiting electron, such that,

$$\frac{mv^2}{r} = \frac{e_0^2}{r^2}, \tag{3.41}$$

where v is the constant speed of the electron and,

$$e_0^2 = \frac{e^2}{4\pi\varepsilon_0}, \tag{3.42}$$

is the force obtained from the potential energy,

$$V(r) = \frac{e^2}{4\pi\varepsilon_0 r}, \tag{3.43}$$

where ε_0 is the permitivity of free the vacuum. In a simple atom such as hydrogen, this potential is interchangeable with the Coulomb force.

The dimensions of ε_0^2 are energy times length. The fundamental constants ε_0^2, $\hbar$ and m are now components of the dimensional analysis. In this approach A will be defined as an arbitrary physical quantity and **dim** A will denote its dimension. The dimensions of *mass*, *length*, and *time* are denoted by M, L, and T. The dimension A is then given by,

$$\mathbf{dim}\, A = M^{\alpha} L^{\beta} T^{\gamma}, \tag{3.44}$$

where α, β,and γ are definition exponents [West88].

---

[17] The coupling force of electromagnetism produces the macro effects of everyday life. If this coupling strength were very different, the world around us would be equally very different. If the electromagnetic force were much strong than the strong nuclear force, than the electrostatic repulsion between protons would overcome the nuclear attractive force and the nucleus containing more than one proton would break up. The world would then be composed of only hydrogen and hydrogen isotopes and possibly large hydrogen molecules. With a weak electromagnetic force, electrons would be free to form a plasma and light would be constantly emitted and absorbed by the charged particles of the plasma. Any light traveling through this plasma would be scattered and distorted.

If the electromagnetic force were very strong then it would be impossible to observe the world through our senses. With a very strong electromagnetic force light traveling through space would *scatter* from other light preventing the image of an object from reaching our eyes or sensors undisturbed.

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In this analysis the expression,

$$e_0^x m^y \hbar^z \,, \tag{3.45}$$

will be defined as a *length* with x, y, and z as unknowns to be solved through dimensional analysis. For the fundamental constants m, e_0 and $\hbar$,

$$\mathbf{dim}\; m = M \,, \tag{3.46}$$

$$\mathbf{dim}\; e_0 = M^{1/2} L^{3/2} T^{-1} \,, \tag{3.47}$$

$$\mathbf{dim}\; \hbar = M L^2 T^{-1} \,. \tag{3.48}$$

Since Eq. (3.45) is a length, the unknown exponents are determined by,

$$\mathbf{dim}\; e_0^x m^y \hbar^z = M^{(x/2)} L^{(3x/2)} T^{-x} M^y M^z L^{2x} T^{-z} = L \tag{3.49}$$

Solving for the three unknowns can be done using the following,

$$\begin{aligned} x + z &= 0, \\ x/2 + y + z &= 0, \\ 3x/2 + 2z &= 1, \end{aligned} \tag{3.50}$$

whose solutions are,

$$\begin{aligned} x &= -2, \\ y &= -1, \\ z &= 2. \end{aligned} \tag{3.51}$$

The characteristic length a_0 is now given by,

$$a_0 = \frac{\hbar^2}{m e_0^2} \,, \tag{3.52}$$

which for the hydrogen atom, a single electron orbiting in an electromagnetic field of the nucleus, is the Bohr radius. The characteristic energy of the electron can be determined in a similar manner to give,

$$E_0 = \frac{m e_0^4}{\hbar^2} \,, \tag{3.53}$$

When values are inserted into Eq. (3.52) and Eq. (3.53),

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$$a_0 = \frac{\hbar^2}{me_0^2} = 0.525 \times 10^{-10}\, m, \tag{3.54}$$

$$E_0 = \frac{me_0^4}{\hbar^2} = 27.2eV. \tag{3.55}$$

If the speed of light is added to the dimensional analysis, the *rest energy* of the orbiting electron can be derived as,

$$E = mc^2. \tag{3.56}$$

The ratio between the characteristic energy and the rest energy is then,

$$\frac{E_0}{mc^2} = \alpha^2, \tag{3.57}$$

which is a dimensionless constant called the *coupling constant* or *fine structure constant*,

$$\alpha_{\text{electromagnetic}} = \frac{e_0^2}{\hbar c} \cong \frac{1}{137.0359895(61)}. \tag{3.58}$$

This constant relates the strength of the electromagnetic field in dimensionless units as the relative strength of the appropriate force between two protons at a separation distance of $10^{-13}\,cm$ [West88]. The significance of the constant is that it allows the coupling between the electromagnetic field and charged particles to be treated as a small perturbation that could be approximated in a series expansion [Davi79], [Hugh91], [Kaku93].

The coupling constants for *weak*, *strong* and *gravitational* interactions vary tremendously. The weak coupling constant is approximately:

$$\alpha_{\text{weak}} \cong \sqrt{2}g^2 \big/ 8m_{\text{proton}}^2 \cong 1.02 \times 10^{-5}, \tag{3.59}$$

where the strong force's interaction is approximately,

$$\alpha_{\text{strong}} = g_s^2 \big/ 4\pi\hbar c \cong 1. \tag{3.60}$$

The gravitational force is significantly weaker with a value of

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$$\alpha_{\text{Newton}} \cong G_N m_p^2 \cong 5.9\times10^{-39} \big/ m_{\text{proton}}^2 \,.\text{[18]} \tag{3.61}$$

With these developments, the description of the electromagnetic force is largely complete. Maxwell's equations can now be combined with the coupling constant to describe the forces between charged particles. When quantum mechanics is added to produce Quantum Electrodynamics

§3.8. CONTINUITY EQUATIONS

In order for Maxwell's equations to properly describe electromagnetic process an expression for the *conservation of charge* must be developed. This will be done through the *Divergence Theorem* or Gauss's Theorem [Byro69]. Consider a stationary volume V, with a surface area S, containing an electric charge with density ρ which is moving with velocity v. The total charge contained in the volume V is given by,

$$\int_V \rho dV \,, \tag{3.62}$$

[18] At *normal* energy levels the coupling constants for the electromagnetic, weak and strong forces are given above. It is believed that at energy levels found during the first instants of the creation of the universe — *The Big Bang* — the coupling constants for these three forces were the same value and there existed a *single* universal force. When the universe cooled the three forces *condensed* and the coupling constants become what we measure them to be today [Kaku93].

Using the coupling constants provided above, the relative force strengths can be described, using dimensional analysis [Isha89]. The gravitational force between two objects whose masses are m_1 and m_2 which are a distance r apart is $F = Gm_1m_2 \big/ r^2$ where the gravitational constant is $G = 6.67\times10^{-11} Nm^2kg^{-2}$. The gravitational constant, G, which sets the scale of gravitational forces, the velocity of light, c, which is assumed to be constant throughout the universe and the electrical charge, e, which is also assumed to be constant throughout the universe — form the fundamental units of physics. These fundamental constants can be used to form *dimensionless* ratios or ratios with only the dimension of length.

The *Fine Structure* constant can be given again as, $e^2 \big/ \hbar c \approx 1\big/137$. The ratio of the gravitational force to the electromagnetic force in a hydrogen atom is $Gm_1m_2 \big/ e^2 \approx 10^{-40}$, which rules out any gravitational influence at the microphysics level. The gravitational constant, $\hbar$ and c form a dimensionless constant $\sqrt{\left(G\hbar \big/ c^3\right)} \approx 10^{-33}\,\text{cm}$, which is called the *Planck* scale. At this distance, the quantum effects of gravity become important. However the energy required to observe this effect is $\approx 10^{19}\text{Gev}$, which is four (4) orders of magnitude beyond which the laws of physics are current understood to be exact — which is the limit of the so–called *Theory of Everything* [Clos83].

so that the time rate of change of this total charge is now,

$$\frac{\partial}{\partial t}\int_V \rho dV \; . \qquad (3.63)$$

The total charge leaving the volume V per unit time is given by,

$$\frac{\partial}{\partial t}\int_V \rho dV = \int_V \nabla\cdot(\rho v)\, dV \; , \qquad (3.64)$$

where the *divergence theorem* of Eq. (3.3) is used for the right side of this expression. The conservation of charge requires that,

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_V \nabla\cdot(\rho v)\, dV = 0 \, , \qquad (3.65)$$

or,

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v)\right] dV = 0 \, . \qquad (3.66)$$

For this expression to hold for an arbitrary volume V, it must be true that,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \, . \qquad (3.67)$$

When a charge ρ moves with velocity v a current is generated such that $\mathbf{j} = \rho v$, which produces an equation of continuity or conservation of charge as,

(VI) $$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \, . \qquad (3.68)$$

This concept allows ρ and j to be considered as sources on the electromagnetic field so that Eq. (I) – (IV) determine the fields produced by a given system of charges and currents. This is the *radiation* view where ρ and $\mathbf{j}$ specify the properties of the antenna, further developed in later sections.

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§3.9. SUMMARY OF MAXWELL'S EQUATIONS

Maxwell's equations, using the *natural units* notation in which $c = \mu_0 = \varepsilon_0 = 1$, [19] can now be summarized as:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, && \text{(a)} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && \text{(b)} \\ \nabla \cdot \mathbf{B} &= 0, && \text{(c)} \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}, && \text{(d)} \\ F &= q\left(\mathbf{E} + v \times \mathbf{B}\right), && \text{(e)} \\ \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} &= 0 && \text{(f).} \end{aligned} \tag{3.69}$$

These equations describe the propagation of electromagnetic waves through free space, in the absence of the source of the radiated energy. It will be these equations that will form the basis of the ultimate goal of this text, the explanation of the conveyance of the electromagnetic force from the antenna of a radio transmitter to the electrons in the metallic antenna of the receiver.

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[19] One of the consequences of using natural units $\hbar = c = \varepsilon_0 = \mu_0 = 1$ removes the need to develop the value of c from Maxwell's equations. In a generic wave equation $\partial^2\xi/\partial t^2 = \left(1/v^2\right)\left(\partial^2\xi/\partial x^2\right)$, whose general solution is $\xi = f_1(x - vt) + f_2(x + vt)$ for any arbitrary functions of f_1 and f_2. In general form the propagation velocity $1/v^2$ is given by ω^2/k^2, where ω is the wave frequency and k is the wave number [Lind56].

In 1857 Wilhelm Weber (1804–1890) and Rudolph Kohlrausch (1809–1858) measured this propagation constant which appears in Maxwell's equation, when a combined system of electrostatic and electromagnetic units are used (esu and msu). They determined the value to be approximately 3×10^{10} cm/sec. The importance of this measurement, which was very close to observed speed of light, was first noted by Gustav Kirchoff (1824–1887). In 1864 Maxwell used this information in his revised electromagnetic theory to assert that electromagnetic waves and light waves are equivalent.

In 1887 Heinrich Hertz (1857–1894) succeeded in generating electromagnetic waves which possessed all the properties of light waves – interference, defraction, and reflection. Joseph Henry observed electrical oscillations and perhaps even propagation of electromagnetic waves as early as 1842 [Hert93].

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In general these equations are difficult to solve, since they require special techniques to deal with the boundary conditions and the coupling between the various terms. In the Cartesian coordinate system, it is laborious to expand the vector algebra and apply the differential operators to the individual terms. This level of detail will be avoided since it is handled so well in other texts [Bala89], [Baru64], [Beke77], [Cott91], [Elli93], [Eyge72], [Jack75], [John88], [Lorr70], [Mari65], [O'rah65], [Pano55], [Ramo84], [Roja71], [Stra41], [Thom85]. The approach taken in later chapters will be to solve the equations for a *source free* radiation field, propagating through free space. The first four Maxwell equations actually represent eight equations — two scalar (divergence) equations, three Cartesian components of the Electric Field's curl equation and three Cartesian components of the Magnetic Field's curl equation. The solution to these eight equations will be six wave equations, three each for the electric and magnetic fields — *the electromagnetic wave equations.*

> *Maxwell's treatment of Electrical Science was differentiated from that of other writers by his insistence on Faraday's conception of electric and magnetic energy as residing in the medium...in this view, the forces acting on electrified...bodies did not form the whole system of forces...but served only to reveal the presence of a vastly more intricate system of forces, which acted through the ether which other material bodies were supposed to be surrounded.*
>
> *— J. H. Jeans [Jean25]*

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