

§14. FOUNDATIONS OF QUANTUM FIELD THEORY

Quantum Field Theory (QFT) is currently the most successful framework for describing interactions between forces and matter at the subatomic level. It provides both theoretical and computational techniques. Its prediction of the interaction between the quantized electromagnetic field and electrons is accurate to within one part in 10^8 . Three of the four forces of nature can be explained with QFT, using the *Standard Model*. Only Gravity remains outside the description of QFT. This chapter will not present the origin or complete details of QFT. These can be found in numerous texts, including: [Bjor64], [Bjor65], [Bobo59], [Chan90], [Itzy80], [Jauc55], [Mand84], [Ramo89], [Ryde85], [Saku67], [Schw61], [Schw58], [Schw94].

QFT is a generalization of the ordinary quantum mechanics of point particles and in principle applies the laws of quantum theory to non-mechanical systems with infinite degrees of freedom. A sub-theory of QFT is Quantum Electrodynamics (QED). QED is the quantum theory of the electron's interaction with the electromagnetic field. There are aspects of QED that will be needed to describe the interaction of the radiated electromagnetic field with the electrons in the conductive metal of the radio antenna. This chapter will lay the foundations for the description of this interaction.

In classical physics such systems are called fields, with the electromagnetic and gravitational fields being examples [Roma69]. A field theory presupposes a continuum of space-time points. This continuum is formally called a differentiable manifold, which can be described by coordinates and to which an affine connection and a metric can be assigned. The specification of the values of all the relevant quantities to each space-time point specifies the configuration of the field. The differential equations constraining the values of the quantities at different space-time points forms the field theory.

§14.1 PROBLEMS WITH QFT

During the early stages in the development of QFT it became clear that there were problems in the description of the electron itself. Even Lorentz's classical theory of the electron produced *infinities* when the radiation produced by the accelerating electron was included in the forces acting on the electron [Lore52].

These *self energies* were first addressed by Dirac in 1927 [Dir27]. Dirac wrote...

...hardly anything has been done up to the present on QED. The questions of the correct treatment of a system in which the forces are propagated with the velocity of light instead of instantaneously, of the production of an electromagnetic field by a moving electron and of the radiation of this field on the electron have not yet been touched.

Dirac combines special relativity and quantum mechanics to create QED. This theory correctly described spin and the magnetic moment of the electron and provided the relativistic corrections to the spectra of hydrogen atoms. This theory also predicted anti-matter which was discovered in 1932 by Anderson [And33].

However other problems began to occur. When higher order corrections to the theory were applied the *infinities* reappeared [Schw94]. These divergences in the theory lay deep within the structure of QED. QED contained integrals which diverged as $x \rightarrow 0$ or in the momentum representation $k \rightarrow \infty$. These divergences were equivalent to Lorentz's *self energy* problem with the electron.

Over the next several decades the technique of *renormalization* was used to hide the problem. Although useful work was done there was always a nagging question that these corrections were masking the true problem — that the theory did not correctly describe the workings of nature. In 1949 Tomonaga, Schwinger and Feynman formulated the solution to these divergences for which they shared the Nobel Prize. In the early 1960's attempts to unify the weak and strong forces lead to new divergences [Pais86], [Crea86].

The quantum field theory developed in the 1930's [Dir27] relied on the *quantization* of the classical field. Although this approach worked for the electromagnetic field, after the addition of perturbation techniques, it failed to work for the *strong* force field of interacting nucleons. A new approach was taken in the 1950's which *generalized* field theory with the abandonment of perturbations and the underlying Lagrangian and instead introduced a set of *axioms* for the quantum field.

§14.2 SIMPLE APPROACH TO QFT

It is the former description of the quantum field that is presented here, one in which the electromagnetic field has been quantized from the

classical description, using the Fourier expansion of the field's vector potential.

In the previous sections quantum mechanics was used to describe a system in which the potential energy is known. However such a theory has limitations that prevent it from being a complete description of nature. Two shortcomings are its incompatibility with special relativity and its inability to describe systems where the number of particles is not a constant. A relativistic quantum field theory has been developed to remedy these problems. This monograph will not make use of this theory but rather present a non-relativistic description based on Hamiltonian mechanics.

Like the classical Hamiltonian formalism in classical mechanics, the quantum mechanical Hamiltonian is not so much a specific equation, but a framework for quantum mechanics in general.

The quantum mechanical form of the Hamiltonian can be construed to describe the interaction between a radiation field and a charged particle in much the same manner as the interaction between a charged particle and an assemblage of particles moving with the speed of light. ^[1]

§14.3 MECHANICAL ANALOGY

One method for developing the mathematics of quantum field theory is by quantizing a set of independent harmonic oscillators. This technique can be simplified by considering a one-dimensional continuous string composed of N point masses connected by *springs*. ^[2] The string can be

¹By deriving the quantum mechanical representation of the electromagnetic field from the Hamiltonian least action principal, it is sufficient to represent the field through the electromagnetic potential \mathbf{A} . This potential representation allows for gauge invariance to be utilized during the quantization process. When the field interaction is quantized, the matter which interacts with the field is represented by a probability amplitude, dependent on space and time coordinates. The probability of finding an electron at a point is $|\psi|^2$. Only this *modulus* can be measured, but its phase is indeterminate. In Quantum Electrodynamics now acquires a new property of symmetry and invariance. This invariance is equivalent to the conservation of charge. This property produces an important behavior. This phase invariance of the matter field can be attributed to a *local* invariance dependent on the point in space-time where it occurs — if and only if the quantum field of electrons is coupled to a quantum field of interaction obeying the same gauge invariance as the classical electromagnetic potential [Tann93]

²This method of developing the quantum field description using a mechanical model has a long history [Bril46]. The first effort of analyzing a one-dimensional vibrating lattice

envisioned as *lumpy* with a total length of L containing N points each with equal mass m coupled with a spring constant k . If each point in this lumpy string is capable of being displaced around its equilibrium in a periodic manner by a distance l , then the string can be constrained to oscillate with a total length of $L = Nl$.

was done by Newton in his attempt to derive a formula for the velocity of sound [Cajo62]. Newton assumed sound was propagated in air in the same manner an elastic wave was propagated along a group of point masses. He assumed the simplest set of masses connected by a restoring force shown in the same manner as in Figure 5.0. Neighboring masses were assumed to *attract* each other with a elastic force e . Using m as the mass of each of the particles and d to be the distance between each particle when it was in a state of equilibrium, the velocity of propagation V of an elastic wave was given by Newton as, $V = d\sqrt{e/m} = \sqrt{ed/\rho}$, where ρ is the density of the air surrounding the particles. Using ed as the isothermal modulus of air, the figure Newton computed was less than the experimental value. In 1822 Laplace pointed out that the expansions and contractions of air took place adiabatically and the adiabatic elastic constant should be used instead of the isothermal. The calculation using the proper constant produced an excellent agreement with the experimental data.

Using this approach to describe the *vibrations* in elastic media continued with John and Daniel Bernoulli. Using Bernoulli's formulation of the supposition principal, which states that the general motion of a vibrating system is given by a superposition of its proper vibrations, the beginnings of theoretical physics became distinct from mechanics. Lagrange and Euler continued to develop — with much controversy — the work of the Bernoulli's through the end of the eighteenth century. At that point the basis for theoretical physics and modern mathematical analysis was laid. The theory of proper functions, Fourier expansions, partial differential equations, wave propagation and the atomic theory of solids and crystal structures were first associated with the vibrations in strings.

In 1830 Cauchy used Newton's model in an attempt to describe the dispersion of optical waves. Cauchy assumed light waves were just high frequency elastic waves. The length of the waves was estimated to be comparable with the distance between the point masses of a crystal lattice and thus their velocity was assumed to be independent of the wavelength.

Using theory of a vibrating string, quantum mechanics provides a set of rules which can be applied in a general way to mechanical systems. The application of these rules to classical fields results in a quantum field theory. However elementary (meaning non-relativistic) quantum mechanics is not consistent when it is combined with classical field theory unless the classical electromagnetic field is also quantized [Heis30], [Hene62], [Schw58]. Following the development technique of quantum mechanics, quantum field theory is linked to classical field theory through the correspondence principle. Using this principle, the excitations of the quantum field behave as if they were particles. In the case of the electromagnetic field the particle is a photon. By using a mechanical model containing point particles, connected by a linear restoring force, a visualization of the field can be constructed. The extension of this mechanical model to a quantum field theory has several limitations, but this will not be important for the description of the quantized electromagnetic field.

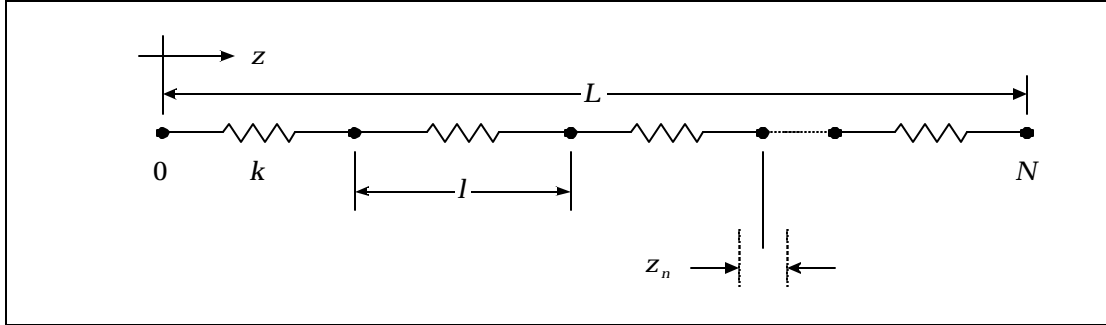


Figure 5.0 — An ideal string of length L with N point masses. Each mass is connected by a *spring* with a spring constant k . Each point mass is constrained to oscillate around its equilibrium point by a distance l . This displacement is given by the function $\phi_i(z, t)$

If the point masses are constrained to vibrate in only one direction, z , then the displacement of the n th particle can be denoted by z_n . The kinetic energy of all the point particle can be given by:

$$T = \sum_{n=0}^{N+1} \frac{1}{2} m \dot{z}_n^2. \quad (14.1)$$

It is assumed that there is potential energy associated with the distortions of the string and that this, to the lowest order in the z_n direction is given by:

$$V = \sum_{n=0}^{N+1} \frac{1}{2} (kN/L) (z_n - z_{n-1})^2, \quad (14.2)$$

with the constraint $z_0 = z_{N+1} = 0$.

The classical action for this mechanical system is given by:

$$S = \int_{t_a}^{t_b} \sum_{n=1}^{N+1} \left[\frac{1}{2} m \dot{z}_n^2 - \frac{1}{2} (kN/L) (z_n - z_{n-1})^2 \right] dt. \quad (14.3)$$

If the discrete system is extended to a continuum the individual displacement can be described by a function of both position and time. If the function ϕ_n is the displacement of the n th oscillator from its equilibrium point then the boundary constraints are $\phi_0 = \phi_N$ and $d\phi_0/dt = d\phi_N/dt$. The kinetic and potential energies of the string are then given by,

$$T = \frac{1}{2} m \sum_{n=0}^{N-1} \left(\frac{d\phi_n}{dt} \right)^2, \quad (14.4)$$

and

$$V = \frac{1}{2} k \sum_{n=0}^{N-1} (\phi_n - \phi_{n+1})^2. \quad (14.5)$$

By expanding the number of points in the discrete string to a large number a continuous system can be developed. If $N \rightarrow \infty$ and $l \rightarrow 0$ and the length of the string remains fixed, such that $L = Nl$ and the mass per unit length is m/l and the string tension is kl , then the displacement of the string in the z direction and energy of the string can be given as a *continuous field* $\phi(z, t)$, where,

$$\phi_n(t) = \phi(z_n, t) \rightarrow \phi(z, t), \quad (14.6)$$

giving the kinetic and potential energy as,

$$T = \frac{1}{2} m \sum_{n=0}^{N-1} l \left(\frac{d\phi_n}{dt} \right)^2 \rightarrow \frac{1}{2} \frac{m}{l} \int_0^L \left(\frac{\partial \phi(z, t)}{\partial t} \right)^2 dt, \quad (14.7)$$

$$V = \frac{1}{2} kl \sum_{n=0}^{N-1} l \left(\frac{\phi_n - \phi_{n+1}}{l} \right)^2 \rightarrow \frac{1}{2} kl \int_0^L \left(\frac{\partial \phi(z, t)}{\partial z} \right)^2 dz. \quad (14.8)$$

The Lagrangian and Hamiltonian of the continuous string can now be constructed using these expressions.

$$L = T - V = \int_0^L \left\{ \frac{1}{2} \frac{m}{l} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} kl \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} dz = \int_0^L \mathcal{L}(z, t) dz, \quad (14.9)$$

$$H = T + V = \int_0^L \left\{ \frac{1}{2} \frac{m}{l} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} kl \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} dz = \int_0^L \mathcal{H}(z, t) dz. \quad (14.10)$$

where \mathcal{L} and \mathcal{H} are the Lagrangian densities.

The previous expressions can be placed in the standard form of the Euler-Lagrange equations by absorbing the spring constant kl through the transformation $\phi \rightarrow \sqrt{kl} \phi$ and the introduction of the wave velocity $v^2 = kl^2/m$, giving the Lagrangian and Hamiltonian as,

$$\mathcal{L} = \frac{1}{2} \int_0^L \left\{ \frac{1}{v^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} dz = \mathcal{L} \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial z} \right), \quad (14.11)$$

$$H = \frac{1}{2} \int_0^L \left\{ \frac{1}{v^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} dz. \quad (14.12)$$

The equations of motion for the string are now given using the *principal of least action* [Gold51]. Using the Lagrangian principles developed previously the least action principle can be stated as,

$$\delta S = \delta \int L \left(\phi_i, \frac{\partial \phi_i}{\partial t}, t \right) = 0, \quad (14.13)$$

giving the Euler–Lagrange equations of motion as,

$$\frac{d}{dt} \frac{\partial L}{\partial \left(\frac{d\phi_i}{dt} \right)} - \frac{\partial L}{\partial \phi_i} = 0. \quad (14.14)$$

In order to proceed with the quantization of the field equations, the discrete oscillators describing the string's motion — and eventually describing the electromagnetic field — will be increased to infinity by allowing $N \rightarrow \infty$ and $l \rightarrow 0$ such that the Euler–Lagrange equations can be written as,

$$\frac{d}{dt} \frac{\partial L}{\partial \left(\frac{d\phi_i}{dt} \right)} - \frac{\partial L}{\partial \phi_i} \xrightarrow{\text{small } l} l\sqrt{kl} \left\{ \frac{\partial}{\partial t} \frac{\partial L}{\partial \left(\frac{\partial \phi}{\partial t} \right)} + \frac{\partial}{\partial z} \frac{\partial L}{\partial \left(\frac{\partial \phi}{\partial z} \right)} \right\}, \quad (14.15)$$

with the term $l\sqrt{kl}$ being discarded.

The result of this *mechanical* example is the basis for quantized the electromagnetic field. The concept of a continuous field can be developed to describe a system containing an infinite number of particles. In the mechanical model the field equations $\phi(z, t)$ describes the *displacement* of each particle from its normal position $z \pm \frac{1}{2}l$. Although a mechanical example was used these concepts can be generalized to describe any field which is *displacement* of some dynamical system.

The next step in *quantizing* the mechanical model is to develop the solutions to the wave equation which satisfies the boundary conditions,

$$\phi \approx e^{\pm i(k_n z - \omega_n t)}, \quad (14.16)$$

where the oscillators are periodically spaced with,

$$k_n = \frac{2\pi n}{L}, \quad (14.17)$$

which gives the wave equation of,

$$\frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (14.18)$$

where,

$$v^2 = \frac{\omega_n^2}{k_n^2}. \quad (14.19)$$

Since both positive and negative frequencies are allowed in the wave equation, the following notation will be used to distinguish one from the other. For positive frequencies, $+\omega_n$ the solution to the wave equation can be written as,

$$\phi_n(z, t) = \frac{1}{\sqrt{L}} e^{i(k_n z - \omega_n t)}, \quad (14.20)$$

and for negative frequencies, $-\omega_n$ the solution can be written as,

$$\phi_n(z, t) = \frac{1}{\sqrt{L}} e^{i(k_n z + \omega_n t)}. \quad (14.21)$$

The expression for the negative frequencies represents the complex conjugate of the positive frequencies and will be written as $\phi_n^*(z, t)$. The normalization of these n frequency states is given by,

$$\int_0^L \phi_n(z, t) \phi_m^*(z, t) dz = \delta_{n,m}, \quad (14.22)$$

which can be rewritten as,

$$\int_0^L \phi_n(z, t) \phi_m(z, t) dz = \delta_{n,-m} e^{-2i\omega_n t}. \quad (14.23)$$

The field can now be expanded in a manner similar to Eq. (10.13),

$$\phi(z, t) = \sum_{n=-\infty}^{\infty} c_n \{ a_n(0) \phi_n(z, t) + a_n^*(0) \phi_n^*(z, t) \}, \quad (14.24)$$

where $a_n(0)$ are the coefficients of each normal mode in the expansion. and c_n is a real normalization factor. The time dependence of each normal mode can be incorporated by,

$$a_n(t) = a_n(0) e^{-i\omega_n t}, \quad (14.25)$$

resulting in,

$$\phi(z, t) = \sum_{n=-\infty}^{\infty} c_n \{ a_n(t) e^{i k z} + a_n^*(t) e^{-i k z} \}. \quad (14.26)$$

The final step is to write the equation for the harmonic oscillator for each point in the mechanical system as,

$$\ddot{a}_n(t) + \omega_n^2 a_n(t) = 0. \quad (14.27)$$

By quantizing the harmonic oscillator the transition from classical field theory to quantum field theory will be made. Since the Lagrangian approach to the dynamics of the field has been previously used, the energy of the mechanical systems dynamical variables $a_n(t)$ can be evaluated. By rewriting the orthonormal relations in Eq. (14.22) as,

$$\int_0^L a_n(0) \phi_n(z, t) a_n^*(0) \phi_m^*(z, t) dz = \delta_{n,m} |a_n(0)|^2 = \delta_{n,m} |a_n(t)|^2, \quad (14.28)$$

and

$$\int_0^L a_n(0) \phi_n(z, t) a_n(0) \phi_m(z, t) dz = \delta_{-n,m} a_n(0) a_{-n}(0) e^{-2i\omega_n t} = \delta_{-n,m} a_n(t) a_{-n}(t), \quad (14.29)$$

allows the kinetic energy to be written as,

$$\begin{aligned} T &= \frac{1}{2v^2} \int_0^L \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} dz, \\ &= \frac{1}{2v^2} \sum \left[2c_n^2 |\dot{a}_n(t)|^2 + c_n c_{-n} \dot{a}_n(t) \dot{a}_{-n}(t) + c_n c_{-n} \dot{a}_n^*(t) \dot{a}_{-n}^*(t) \right]. \end{aligned} \quad (14.30)$$

and the potential energy to be written as,

$$\begin{aligned}
 V &= \frac{1}{2v^2} \int_0^L \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} dz, \\
 &= \frac{1}{2} \sum_{n=-\infty}^{\infty} k_n^2 \left[2c_n^2 |a_n(t)|^2 + c_n c_{-n} a_n(t) a_{-n}(t) + c_n c_{-n} a_n^*(t) a_{-n}^*(t) \right]
 \end{aligned} \tag{14.31}$$

By equating $\dot{a}_n(t) = -i\omega_n a_n(t)$ the Hamiltonian of the dynamical system can be written as,

$$\begin{aligned}
 H &= T + V, \\
 &= \sum_{n=-\infty}^{\infty} \left\{ c_n^2 \left(\frac{\omega_n^2}{v^2} + k_n^2 \right) a_n^*(t) a_n(t) + \dots \right. \\
 &\quad \left. \dots + \frac{1}{2} c_n c_{-n} \left(-\frac{\omega_n^2}{v^2} + k_n^2 \right) [a_n(t) a_{-n}(t) + a_n^*(t) a_{-n}^*(t)] \right\}, \\
 &= \sum_{n=-\infty}^{\infty} c_n^2 \frac{2\omega_n^2}{v^2} a_n^*(t) a_n(t).
 \end{aligned} \tag{14.32}$$

By using *natural* units in which the frequency ω has units of energy, the normalization factor c_n can be chosen such that, $c_n = \sqrt{v^2/2\omega_n}$ which simplifies the Hamiltonian to,

$$H = \sum_{n=-\infty}^{\infty} \omega_n a_n^*(t) a_n(t). \tag{14.33}$$

§14.3.1 Canonical Coordinates of the String

By selecting an alternative set of coordinates the Fourier coefficients of the Hamiltonian can be expressed as generalized position and momentum. The Lagrangian of the dynamical system allows the generalized position to be given as,

$$q_n(t) = \frac{1}{\sqrt{2\omega_n}} [a_n(t) + a_n^*(t)], \tag{14.34}$$

and the generalized momentum to be given as,

$$p_n(t) = \frac{1}{\sqrt{2\omega_n}} [a_n(t) - a_n^*(t)] = \frac{dq_n(t)}{dt}. \tag{14.35}$$

which now allows the Fourier coefficients to be expressed in terms of a coordinate system,

$$a_n = \frac{ip_n + \omega_n q_n}{\sqrt{2\omega_n}}, \quad (14.36)$$

and,

$$a_n^* = \frac{-ip_n + \omega_n q_n}{\sqrt{2\omega_n}}. \quad (14.37)$$

Using the Fourier coefficients the Hamiltonian can be rewritten as,

$$\mathbb{H} = \sum_{n=-\infty}^{\infty} \frac{1}{2} [p_n^2 + \omega_n^2 q_n^2]. \quad (14.38)$$

This expression of the Hamiltonian is simply the sum of the Hamiltonians of each independent harmonic oscillator in the string. Hamilton's equations of motion can now be stated from the previous development as,

$$\dot{q}_n = \frac{\partial \mathbb{H}}{\partial p_n} = p_n, \quad (14.39)$$

and,

$$\dot{p}_n = -\frac{\partial \mathbb{H}}{\partial q_n} = -\omega_n^2 q_n. \quad (14.40)$$

which is the equations of motion for an uncouple oscillator.

§14.3.2 Quantizing the Mechanical System

Using the canonical variables developed above, a transformation from their Poisson bracket form into a commutator form can be performed using the techniques previously developed [Sch68]. For generalized coordinates and momenta ^[3], these canonical variables are related by,

³The process of treating the generalized coordinates p_n and q_n as quantized variables is called *first quantization*.

$$[q_n, p_n] = i\hbar\delta_{n,m}, \quad (14.41)$$

and

$$[q_n, q_n] = [p_n, p_n] = 0. \quad (14.42)$$

Using *natural* units $\hbar = c = 1$ these variables can be restated as,

$$[a_n, a_n^*] = \delta_{n,m}, \quad (14.43)$$

and,

$$[a_n, a_m] = [a_n^*, a_m^*] = 0. \quad (14.44)$$

The field equations can now be restated as,

$$\phi(z, t) = v \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\omega_n L}} \left\{ a_n e^{i(k_n z - \omega_n t)} + a_n^* e^{-i(k_n z - \omega_n t)} \right\}. \quad (14.45)$$

The Hamiltonian of the string can be written as,

$$\begin{aligned} H &= \sum_{n=-\infty}^{\infty} \frac{\omega_n}{2} [a_n^* a_n + a_n a_n^*], \\ &= \sum_{n=-\infty}^{\infty} \omega_n \left[a_n^* a_n + \frac{1}{2} \right]. \end{aligned} \quad (14.46)$$

§14.4 CANONICAL MOMENTUM OF THE STRING

The commutation rules between a and a^* imply a relationship between the fields ϕ . If the ϕ field is considered a canonical coordinate, then using the Lagrangian defined in Eq. (14.11) can be used to define the canonical momentum as,

$$\Pi(z, t) = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial t} \right)} = \frac{1}{v^2} \frac{\partial \phi}{\partial t}. \quad (14.47)$$

By generalizing the commutation rules given in Eq. (14.41) and Eq. (14.42) to the continuous fields coordinates ϕ and Π ,

$$[\phi(z, t), \Pi(z', t)] = i\delta(z - z'), \quad (14.48)$$

$$[\Pi(z, t), \Pi(z', t)] = 0, \quad (14.49)$$

$$[\phi(z, t), \phi(z', t)] = 0. \quad (14.50)$$

It is useful at this point to demonstrate these commutation relationships explicitly. Expanding the canonical momentum of Eq. (14.47) in terms of the harmonic oscillator's dynamical variable gives,

$$\Pi(z, t) = \frac{1}{v} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\omega_n L}} \left\{ -i\omega_n a_n e^{i(k_n z - \omega_n t)} + i\omega_n a_n^* e^{-i(k_n z - \omega_n t)} \right\} \quad (14.51)$$

Expanding Eq. (14.49) using Eq. (14.51) gives,

$$\begin{aligned} [\Pi(z, t), \phi(z', t)] &= -\frac{i}{2L} \sum_{n,m} \frac{\sqrt{\omega_n}}{\sqrt{\omega_m}} \left\{ \begin{aligned} &[a_n, a_m] e^{i(k_n z + k_m z') - i(\omega_n + \omega_m)t} - \dots \\ &[a_n^*, a_m^*] e^{-i(k_n z + k_m z') + i(\omega_n - \omega_m)t} + \dots \\ &[a_n, a_m^*] e^{i(k_n z - k_m z') - i(\omega_n - \omega_m)t} - \dots \\ &[a_n^*, a_m] e^{-i(k_n z + k_m z') + i(\omega_n - \omega_m)t} \end{aligned} \right\} \\ &= -i \frac{1}{L} \sum_n e^{ik_n(z-z')} \equiv -iI(z, z'). \end{aligned} \quad (14.52)$$

Since the exponential term inside the summation of Eq. (14.52) is complete and orthonormal, it can be expressed in terms of the exponential such that,

$$\int_0^L I(z, z') e^{ik_n z'} dz' = e^{ik_n z}. \quad (14.53)$$

Eq. (14.53) is the definition of the Dirac delta function,

$$I(z, z') = \frac{1}{L} \sum_n e^{ik_n(z-z')} = \delta(z-z'), \quad (14.54)$$

which results in Eq. (14.48).

The commutation relationships in Eq. (14.49) and Eq. (14.50) can be shown as,

$$\begin{aligned} [\phi(z, t), \phi(z', t)] &= \frac{v^2}{2L} \sum_{n,m} \frac{1}{\sqrt{\omega_n \omega_m}} \left\{ \begin{aligned} &[a_n, a_m^*] e^{i(k_n z - k_m z') - i(\omega_n - \omega_m)t} + \dots \\ &[a_n^*, a_m] e^{-i(k_n z - k_m z') + i(\omega_n - \omega_m)t} \end{aligned} \right\} \\ &= \frac{v^2}{2L} \sum_n \frac{1}{\omega_n} e^{ik_n(z-z')} [1-1] \equiv 0. \end{aligned} \quad (14.56)$$

They assert that all originates from the sky and its fires, and that first the fire changes into the breezes of the air and from the water is produced, and from water comes the earth and is reverse order everything reproduces from the earth, first water, next the air and then the heat — these are four elements never cease to interchange with each other, passing from the sky to the earth, and from the earth to the stars of the firmament.

— Lucrezio, *IŞA.C*