§15. QUANTIZING THE CLASSICAL RADIATION FIELD

Up to this point the description of the electromagnetic field using the vector potential $\mathbf{A}$ has simply been a convenience. Although both the vector and scalar potentials were utilized to expand the classical radiation function, their existence has always been *mathematical*. The macroscopic description of the electromagnetic field requires two Euclidean vectors — the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. From these two vectors, the field potential $\mathbf{A}$ can be inferred.

The transition from classical electrodynamics to quantum electrodynamics can be performed through the following processes. Suppose a volume V is divided into a large number N of smaller volumes, with the n^{th} cell specified by the coordinate $\mathbf{r}_n$. The distribution of the electromagnetic filed within the n^{th} volume at time t can be approximated by the potential vectors, $\mathbf{A}(\mathbf{r}_1,t),\ldots,\mathbf{A}(\mathbf{r}_n,t)$.

In order to transform the classical description of the electromagnetic field into a quantum mechanical form, the potential vectors $\mathbf{A}(\mathbf{r}_n,t)$ will be restated as *operators* acting on the vectors of a Hilbert space in such a manner that all the *observables* in the system are Hermitian operators which have *real* eigenvalues. The second step in the quantization process is to allow the number of small volumes to tend to infinity, allowing the system to have an infinite number of degree's of freedom. [1]

The classical description of the fields given by the Hamiltonian formulation provides the transition to the quantum mechanical theory of the electromagnetic field equations. Using the canonical variables which are the generalized coordinates $X_{\mathbf{k},\alpha}$ and the generalized momenta $P_{\mathbf{k},\alpha}$, the scalar potential $\mathbf{A}$ and the $\mathbf{E}$ and the $\mathbf{B}$ fields become quantum field operators.

> *There is no place in this new kind of physics for the field and matter, for the field is the only reality.*
>
> — Albert Einstein [Cape61]

The *quantum operators* $\mathbf{A}(\mathbf{r},t)$ form one of the elements of the mathematical description of the quantum field. They will be referred to as *field variables*. These operators are defined over the infinite dimension Hilbert space, whose algebra is fixed by the commutation rules and by assuming the existence of a particular state vector describing the vacuum state of the system.

As a result of the quantization, the electromagnetic field's potential is,

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}}\sum_{i=1}^{2}\sqrt{\frac{2\pi\hbar c^{2}}{V\omega_k}}\left\{\hat{e}_{k,i}a_{k,i}(t)e^{i\mathbf{k}\cdot\mathbf{r}}+\hat{e}^{*}_{k,i}a^{*}_{k,i}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right\} \tag{15.1}$$

By replacing the classical vector potential $\mathbf{A}(\mathbf{r},t)$ by the linear operator $\hat{\mathbf{A}}(\mathbf{r},t)$, which is a function of the Euclidean coordinates, every physical quantity of the system then becomes a Hermitian operator — also called an observable — whose real eigenvalues represent a precise measurement of this quantity.

For the radiation field using the transverse gauge, $\nabla\cdot\mathbf{A}=0$ and with the absence of external sources, indicated by $\phi=0$, the electric and magnetic fields are given by the no familiar expressions, $\mathbf{E}=-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\nabla\times\mathbf{A}$, where $\mathbf{A}$ satisfies the wave equation,

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t}=0\,, \tag{15.2}$$

resembles the dynamic behavior of a collection of harmonic oscillators. In the quantum formulation of the radiation field, the oscillation system described by Eq. (15.1) has a variety of energy states, each associated with a quantum state. These states can be interpreted in terms of specific quanta.

§15.1 QUANTIZING THE SCHRÖDINGER EQUATION

The classical Lagrangian density which results in the Schrödinger equation is given by,

$$\mathsf{L}=i\hbar\psi^*\frac{d\psi}{dt}-\frac{\hbar^2}{2m}\nabla^2\left(\nabla\psi^*\right)\cdot\left(\nabla\psi\right)-V\left(\mathbf{r},t\right)\psi^*\psi\,, \tag{15.3}$$

where L is not Hermitian. The Euler–Lagrange equation for ψ gives,

$$-i\hbar\frac{d\psi^*}{dt}=-\frac{\hbar^2}{2m}\nabla^2\psi^*+V\left(\mathbf{r},t\right)\psi^* \tag{15.4}$$

and for ψ^*,

$$i\hbar\frac{d\psi}{dt}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\left(\mathbf{r},t\right)\psi \tag{15.5}$$

where these equations are the Schrödinger equation and its complex conjugate. The momentum conjugate for ψ is given by,

$$\pi=\frac{\delta\mathsf{L}}{\delta\dfrac{\partial\psi}{\partial t}}=\frac{\partial\mathsf{L}}{\partial\dfrac{d\psi}{dt}}=i\hbar\psi^* \tag{15.6}$$

Since $\dfrac{d\psi^*}{dt}$ does not occur in Eq. (15.3), ψ^* has no conjugate momentum.

The Hamiltonian density is now given by,

$$\mathsf{H}=\pi\frac{d\psi}{dt}-\mathsf{L}=\frac{i\hbar}{2m}\left(\nabla\pi\right)\cdot\left(\nabla\psi\right)-\frac{i}{\hbar}V\pi\psi \tag{15.7}$$

The canonical equations of motion are now,

$$\begin{aligned}\frac{d\psi}{dt}=\frac{d\mathsf{H}}{\delta\pi}&=\frac{\partial\mathsf{H}}{\partial\pi}-\sum_i\frac{\partial}{\partial x_i}\frac{\partial\mathsf{H}}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)},\\&=-\frac{i}{\hbar}V\psi+\frac{i\hbar}{2m}\nabla^2\psi.\end{aligned}\tag{15.8}$$

$$\begin{aligned}\frac{d\pi}{dt}=\frac{d\mathsf{H}}{\delta\pi}&=\frac{\partial\mathsf{H}}{\partial\pi}-\sum_i\frac{\partial}{\partial x_i}\frac{\partial\mathsf{H}}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)},\\&=-\frac{i}{\hbar}V\pi+\frac{i\hbar}{2m}\nabla^2\pi.\end{aligned}\tag{15.9}$$

§15.2 QUANTIZING THE RADIATION FIELD

§15.2.1 FIELD COMMUTATION MODES

§15.2.2 ZERO POINT ENERGY

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[1] This *limit expression* method provides a simplified approach to quantizing the electromagnetic field, but it does have serious drawbacks. The major problem is that the limit method only considers *local* fields which are described by function of space–time coordinates. It has been shown that this approach is the *only* description of the quantized field.