§5. THE RADIATED FIELD

In the previous section the Lorentz force equation was developed which describes the force felt by a charged particle in the presence of a vector potential field. If this potential field is varying with time, then the charged particle will feel the oscillating force, which will impart a momentum to the charged particle. If the particle is free to move, as a free electron is in the outer valance shell of a metal, than this motion will cause a current to be generated, which in turn can be amplified and transformed into a signal used by a radio receiver. Some additional work needs to be done though before this phenomenon can be fully explained.

Given the *time periodic* current $\mathbf{j}(\mathbf{r},t)$, the radiated field potential given in Eq. (4.23) and the traveling wave equation in Eq. (4.30), what is of interest is the field potential equation at some distance $\mathbf{r}$ from the source of the time periodic current.

From Maxwell's equations it follows that all the quantities that enter into them depend on $\mathbf{r}$ and t. This time dependence can be expressed in a complex exponential notation by writing for any such function $F(\mathbf{r},t)=F(\mathbf{r})e^{-i\omega t}$. [1]

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[1] The mathematics of *periodic* functions can be greatly simplified by the use of the exponential notation [Lorr70], [Chen83a]. The instantaneous time–dependent expression of a sinusoidal scalar quantity, such as the current flowing in a wire can be written as either a cosine or sine function. Given a function of the form $f(x)=f(x_0\cos(\omega t+\theta))$ where x_0 is the amplitude of x, ω is the angular frequency and t is time. The quantity $(\omega t+\theta)$ is the phase, or phase angle with θ being the phase at time $t=0$. The first and second derivatives of the function $f(x)$ are found in the equations of electrodynamics. The evaluation of these derivatives are simplified by using the exponential notation. By noting that $e^{i\omega t}=\cos\omega t+i\sin\omega t$, then $x=x_0\cos\omega t=\mathrm{Re:}\{x_0e^{i\omega t}\}$. This notation can be simplified by ignoring the Re: operator such that $x=x_0e^{i\omega t}$ and the derivatives are then given as, $dx/dt=i\omega x_0e^{i\omega t}=i\omega x$ and $d^2x/dt^2=(i\omega)^2x_0e^{i\omega t}=(i\omega)^2x$, In this convention the operator d/dt is replaced by the factor $i\omega$.

The coefficient before the exponential function can also be complex and be represented as $\mathrm{Re:}\{ix_0e^{i\omega t}\}=-x_0\sin\omega t$ if x_0 is real. Since the imaginary number can be defined as $i=e^{i\omega t}$, $\mathrm{Re:}\{ix_0e^{i\omega t}\}=\mathrm{Re:}\{ix_0e^{i(\omega t+\pi/2)}\}=x_0\cos(\omega t+\pi/2)=-x_0\sin\omega t$.

The most common use of this notation is in the description of plane wave propagation. If a quantity α is propagating with a velocity u is defined at $z=0$ by $\alpha=\alpha_0\cos\omega t$ then for any position z in the direction of propagation of the plane wave $\alpha=\alpha_0\cos\omega(t-z/u)$.

§5.1. PLANE WAVES IN FREE SPACE

The wave equations Eq. (4.30) and Eq. (4.31) describe *vector* waves and can be combined into a simplified *scalar* wave equation if they are represented in rectangular coordinates, such that.

$$\nabla^2\psi - \frac{\partial^2\psi}{\partial t^2} = 0\,, \tag{5.1}$$

for the wave components, $\psi = E_x, E_y, E_z, B_x, B_y,$and B_z .

For any two vector field components that lie in a plane, that is any two components of he vector triple *x, y, z* that are used to form a plane, can be ignored. The remaining component can be used to describe a one – dimensional wave equation,

$$\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial t^2} = 0\,, \tag{5.2}$$

since y and z form the plane and x is considered the *propagation* vector *normal* to the plane (formed by y and z). The general solution to Eq. (5.2) can be found using a combination of arbitrary functions,

$$\psi(x,t) = f\,(x+ct) + g\,(x-ct)\,. \tag{5.3}$$

The function $f\,(x+ct)$ represents a wave traveling in the negative x direction and the function $g(x-ct)$ represents a wave traveling in the positive x direction. By using only the positive propagation direction, the wave equation solution can be expanded as a Fourier series,

$$\psi(x,t) = \sum_n e^{i(\omega_n t - k_n x)}\,. \tag{5.4}$$

This expression describes a wave with constant amplitude α_0. The wave fronts are surfaces of constant phase and are perpendicular to the z–axis. The phase angle $(t - z/u)$ is a constant such that $(t - z/u) = C$ for a point traveling with a phase velocity $dz/dt = u$.

This velocity is the wave's *phase velocity* because it is the velocity with which the phase $\omega[t-(z/u)]$ is propagating in space. using this notation the wave can be rewritten as $\alpha = \alpha_0 \cos(\omega t - kz)$ where $k = \omega/u$ is the *wave number*. When this wave travels in a particular direction specified by the unit vector $\mathbf{n}$, the wave fronts are then *normal* to $\mathbf{n}$ such that the wave is described by, $\alpha = \alpha_0 e^{i(\omega t\;\; k\mathbf{n}\cdot\mathbf{r})}$, where $\mathbf{n}\cdot\mathbf{r} = z$ in the case that the unit vector coincides with the unit vector in the z–direction.

All the possible frequencies ω_n are related to the *wave numbers* (propagation constants) by,

$$\omega_n = k_n c\ . \tag{5.5}$$

For a monochromatic wave, the subscript can be dropped, since a single frequency will be considered. This results in the Fourier expansion of the wave front traveling in the *x* direction of,

$$\psi(x,t) \approx e^{-i(\omega t - kx)}\ . \tag{5.6}$$

To generalize this notation, the propagation *direction* will be denoted by the vector **k**, which allows the form of the wave equation to be maintained. The vector **k** is a direction *normal* to the wave front plane, with a magnitude of $|\mathbf{k}| = k$. For a wave propagating in an arbitrary direction **r**, the expanded Fourier expression is now,

$$\psi(\mathbf{r},t) = e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{5.7}$$

where **k** is the propagation vector and **r** is the propagation direction, relative to the moving plane wave front.

Using this approach the solutions to Maxwell's wave equations can be developed using the Fourier transform of the vector operators. This allows t he replacement of the vector operators for divergence, curl and gradient. Starting with the following definition in Cartesian coordinates of the expanded plane wave:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{\partial}{\partial x}e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{\partial}{\partial y}e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{\partial}{\partial z}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{5.8}$$

the Gradient of the wave is given by,

$$\begin{aligned} \nabla e^{i\mathbf{k}\cdot\mathbf{r}} &= i\,k_x e^{i\mathbf{k}\cdot\mathbf{r}} + ik_y e^{i\mathbf{k}\cdot\mathbf{r}} + i\,k_z e^{i\mathbf{k}\cdot\mathbf{r}}, \\ &= \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{5.9}$$

The Divergence of the wave is given by:

$$\begin{aligned} \nabla\times\left(\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}\right) &= \left(A_z i\,k_y - A_y\,ik_z\right)e^{i\mathbf{k}\cdot\mathbf{r}} + \left(A_x i\,k_z - A_z\,ik_x\right)e^{i\mathbf{k}\cdot\mathbf{r}} + \left(A_y\,ik_x - A_x i\,k_z\right)e^{i\mathbf{k}\cdot\mathbf{r}}, \\ &= \left(\mathbf{k}\times\mathbf{A}\right)e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{5.10}$$

The Curl of the of the wave is given by:

$$
\begin{aligned}
\nabla\cdot\left(\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}\right) &= \frac{\partial}{\partial x}\left(A_x e^{i\,\mathbf{k}\,\mathbf{r}}\right)+\frac{\partial}{\partial y}\left(A_y e^{i\,\mathbf{k}\,\mathbf{r}}\right)+\frac{\partial}{\partial z}\left(A_z e^{i\mathbf{k}\cdot\mathbf{r}}\right),\\
&= ik_x A_x e^{i\mathbf{k}\cdot\mathbf{r}}+ik_y A_y e^{i\mathbf{k}\cdot\mathbf{r}}+ik_z A_z e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad (5.11)\\
& i\mathbf{k}\cdot\mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}.
\end{aligned}
$$

These transformation allow Maxwell's equations to be rewritten in the Fourier space (or momentum space once the quantum mechanical description of the propagating electromagnetic wave has been developed).

The unit vector normal to the wave front is $\mathbf{k}/k$ and ξ is the distance from the origin of the coordinate system to the plane of the propagation such that,

$$\frac{1}{k}(\mathbf{k}\cdot\mathbf{r})=\xi. \tag{5.12}$$

Using this new notation Maxwell's equations become,

$$(\text{Ia})\qquad \nabla\cdot\mathbf{E}=\rho\rightarrow\mathbf{k}\cdot\frac{\partial\mathbf{E}}{\partial\xi}=\rho, \tag{5.13}$$

$$(\text{IIa})\qquad \nabla\cdot\mathbf{B}=0\rightarrow\mathbf{k}\cdot\frac{\partial\mathbf{B}}{\partial\xi}=0, \tag{5.14}$$

$$(\text{IIIa})\qquad \nabla\times\mathbf{B}-\mathbf{j}-\frac{\partial\mathbf{E}}{\partial t}=0\rightarrow\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\xi}-k\frac{\partial\mathbf{E}}{\partial t}=0, \tag{5.15}$$

$$(\text{IVa})\qquad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0\rightarrow\mathbf{k}\times\frac{\partial\mathbf{E}}{\partial\xi}+k\frac{\partial\mathbf{B}}{\partial\xi}=0. \tag{5.16}$$

§5.1.1. LONGITUDINAL PROPAGATION COMPONENTS

The propagating electromagnetic wave can be further separated into longitudinal and transverse components.

If the scalar product of $\mathbf{k}$ is made with Eq. (5.15) the results is,

$$\mathbf{k}\cdot\left(\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\xi}\right)-k\left(\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{k}\right)=0. \tag{5.17}$$

The $\mathbf{k}$ vectors in the first term can be exchanged to allow $\mathbf{k}\times\mathbf{k}$ to vanish so that Eq. (5.17) can be written as,

$$\mathbf{k}\cdot\frac{\partial\mathbf{E}}{\partial t}=0\,, \tag{5.18}$$

Multiplying Eq. (5.18) by dt and the resulting expression from Eq. (5.13) by $d\xi$ and then adding the expressions gives,

$$\mathbf{k}\cdot\left(\frac{\partial\mathbf{E}}{\partial\xi}d\xi+\frac{\partial\mathbf{E}}{\partial t}dt\right)=0\,. \tag{5.19}$$

The sum of the terms inside the parenthesis of Eq. (5.19) is the *total differential* of the electric field vector $\mathbf{E}$, which reduces Eq. (5.19) to,

$$\mathbf{k}\cdot d\mathbf{E}=0\,. \tag{5.20}$$

The same operations can be performed on Eq. (5.16) to produce,

$$\mathbf{k}\cdot d\mathbf{B}=0\,. \tag{5.21}$$

Eq. (5.20) and Eq. (5.21) require that the components of the $\mathbf{E}$ and $\mathbf{B}$ fields that are *normal* to the propagating wave front be constant in both time and space. This implies that the longitudinal components of the wave are *static*. Since such fields do not contribute to the propagation of the wave, they can be eliminated from the wave equations, such that, $\mathbf{E}_{longitudinal}\equiv 0$ and $\mathbf{B}_{longitudinal}\equiv 0$ .This leaves the *transverse* components of the wave to contribute to the propagation, such that the wave equations are now,

$$\frac{\partial^2\mathbf{E}}{\partial\xi^2}-\frac{\partial^2\mathbf{E}}{\partial t^2}=0\,, \tag{5.22}$$

and

$$\frac{\partial^2\mathbf{B}}{\partial\xi^2}-\frac{\partial^2\mathbf{B}}{\partial t^2}=0\,. \tag{5.23}$$

The electric and magnetic field are perpendicular to the direction of propagation as well as being perpendicular to each other. The set of vectors {$\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$} constitute an orthogonal set.

§5.2. ENERGY IN THE RADIATED FIELD

As the electromagnetic field propagates through free space is may encounter charged particles in its path. The motion of these particles will be influenced by the electromagnetic field — gaining or losing energy, depending on the direction of motion with respect to the $\mathbf{E}$ field. The law of conservation of energy requires that if the charged particles gain energy then the energy must have been present in the electromagnetic field prior to is influence on the particles.

For a collection of particles, each with change q and velocity v_i, the Lorentz force *felt* by the particles is,

$$\mathbf{F}_i = q_i\left(\mathbf{E} + v \times \mathbf{B}\right). \tag{5.24}$$

The rate at which the electromagnetic field does work on a single charge is,

$$\mathbf{F}_i \cdot v_i = q_i\mathbf{E} \cdot v_i + \mathbf{B} \cdot v_i = q\mathbf{E} \cdot v_i, \tag{5.25}$$

where $v_i \times \mathbf{B} v_i = 0$ is assumed, where the $\mathbf{B}$ field component does no work since it is perpendicular to the velocity of the charged particles.

The total rate at which the field does work per unit volume is now the sum of all the Lorentz forces felt by all the particles in the volume,

$$dW = q\sum_i \mathbf{E} \cdot v_i = q\sum_i \left[E_x\left(v_i\right)_x + E_y\left(v_i\right)_y + E_z\left(v_i\right)_z\right], \tag{5.26}$$

where $\left(1/n\right)\sum_i \left(v_i\right)_x = v_x$ is the component of the *drift* velocity of the charges and similarly for the *y* and *z* components. The rate at which work is done is now,

$$dW = nq\mathbf{E} \cdot v = \mathbf{E} \cdot \mathbf{j}, \tag{5.27}$$

where $\mathbf{j} = nqv$ is the current density resulting form the motion of *n* charged particles with *average* velocity *v*. The expression $\mathbf{E} \cdot \mathbf{j}$ can now be rewritten as,

$$\mathbf{E} \cdot \mathbf{j} = c^2\left(\nabla \times \mathbf{B}\right) \cdot \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{5.28}$$

Using the familiar vector identity $\nabla \cdot \left(\mathbf{E} \times \mathbf{B}\right) = \mathbf{B} \cdot \left(\nabla \times \mathbf{E}\right) - \mathbf{E} \cdot \left(\nabla \times \mathbf{B}\right)$, gives,

$$\mathbf{E} \cdot \mathbf{j} = c^2\mathbf{B} \cdot \left(\nabla \times \mathbf{E}\right) - c^2\nabla \cdot \left(\mathbf{E} \times \mathbf{B}\right) - \frac{\partial\left(\mathbf{E} \cdot \mathbf{E}\right)}{\partial t}. \tag{5.29}$$

Using Faraday's Law, Eq. (IV) gives,

$$\mathbf{E}\cdot\mathbf{j}=-\nabla\cdot(\mathbf{E}\times\mathbf{B})-\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}). \tag{5.30}$$

If there are not charges present in the volume, $\mathbf{E}\cdot\mathbf{j}=0$, there will be no conversion of the field energy into motion and Eq. (5.30) has the same form as Eq. (VII), which expresses the conservation of charge.

§5.3. POYNTING'S THEOREM

Before proceeding with the development of the radiated field potential, the form of the conservation of energy law is important to understand. This law is often called Poynting's theorem [Poyn20]. In a paper published in 1884, John Henry Poynting (1852–1914) examined what the flow of energy must be in the electromagnetic field when the localized energies are altered. [2] Poynting's work, along with Oliver Heavyside (1850–1925) lead directly to a mathematical representation of the conduction current in terms of decaying displacement without requiring a knowledge of the connection between the electromagnetic field and matter. Poynting wrote in [Poyn20]:

> ... we believe that when it (energy) disappears at one point and reappears at another it must have passed through the intervening space, we are forced to conclude that the surrounding medium contains at least a part of the energy and that it (the medium) is capable of transferring it (the energy) from point to point.

For a single charge q, the rate of *doing work* by the external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is $q\nu\cdot\mathbf{B}$, where ν is the velocity of the charge. The magnetic field does no work, since the magnetic force is perpendicular to the charges velocity vector. For a continuous distribution of charge and current, the total rate of *doing work* by the fields in a finite volume V is $\int_V \mathbf{j}\cdot\mathbf{E}dv$. This power represents a conversion of electromagnetic energy into mechanical and/or thermal energy.

Using Eq. (5.30) the equivalent equations of Eq. (4.4) and Eq. (4.1), rewritten in terms of the scalar and vector potential, become equations for the field in terms of $\mathbf{A}$ only,

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[2] John Henry Poynting made several important contributions to electromagnetic field theory besides his famous *Poynting Theorem*. His model of the electromagnetic field employed the distribution of lines of force in space and the flux of energy between these lines represented the electromagnetic field [Harm82].

$$\mathbf{E} = \frac{i}{k}\nabla(\nabla\cdot\mathbf{A}) + ik\mathbf{A}, \tag{5.31}$$

$$\mathbf{B} = \nabla\times\mathbf{A}\,. \tag{5.32}$$

The Poynting vector representing energy flow for harmonic fields is, [3]

$$\begin{aligned} S &= \frac{c}{4\pi}(\mathbf{E}e^{-\omega t})\times(\mathbf{B}e^{-\omega t}), \\ &= (\mathbf{E}\cos\omega t + i\mathbf{E}\sin\omega t)\times(\mathbf{B}\cos\omega t + i\mathbf{B}\sin\omega t). \end{aligned} \tag{5.33}$$

The physical meaning of the Poynting vector is that the time rate of change of electromagnetic energy within a certain volume, plus the energy flowing out through the boundary surfaces surround the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume. This is the statement of the conservation of energy.

Multiplying this express out gives four terms: one each involving $\sin^2\omega t$ and $\cos^2\omega t$ and two involving $\sin\omega t$ and $\cos\omega t$. These last two terms oscillate in time and their average value over time is zero, where the first two terms average to ½

The time average Poynting vector is given by,

$$\overline{S} = \frac{c}{8\pi}\left\{(\mathbf{E}\times\mathbf{B}) + i(\mathbf{E}\times\mathbf{B})\right\}, \tag{5.34}$$

which is equivalent to,

$$\overline{S} = \frac{c}{8\pi}(\mathbf{E}\times\mathbf{B}^*), \tag{5.35}$$

---

[3] From the system of Maxwell's equations (I) to (IV) it is possible to derive a very important expression recognized as *the energy principle in the electromagnetic field.* This principle states that the energy liberated per second in a certain volume shall pass through a close surface surrounding the volume according to the following, $-\frac{d}{dt}\int\left\{u_{el} + u_m\right\}dv = \int\psi dv + \oint S_n dA$, where, $u_{el} = \frac{1}{4\pi}\int\mathbf{E}\cdot d\mathbf{D}$ is the electric field density, $u_m = \frac{1}{4\pi}\int\mathbf{H}\cdot d\mathbf{B}$ is the magnetic field density and $\psi = \mathbf{j}\cdot\mathbf{E}$ represents the work from the field expended upon the electric current density. $S_n dA$ gives the energy passes per second through the surface element dA, in the direction of its normal **n.** The term S is called the *Poynting vector*.

where $\mathbf{B}^*$ is the complex conjugate of $\mathbf{B}$.

§5.4. VECTOR POTENTIAL DESCRIPTION OF THE RADIATED FIELD

Potential problems are, in general, of two types – those in which the charge distribution is known and the goal is to determine the resulting potential. These problems find their solutions in Poisson's equation. The second type are those in which a potential must be determined which satisfies given boundary conditions. The general solution to the second type of problem depends on the geometry of the surfaces on which the boundary conditions are given. In such solutions, it is necessary to find a system of coordinates in which the potential equation *separates*, in which a solution may be found which is a product of functions of the separate variables. This separation of variables is appropriate for the problems in which the boundary surfaces are parametric surfaces, that is, surfaces in which one of the coordinates is constant. Spherical polar coordinate are suitable for problems involving radiation field potentials in free space.

The field vector potential,

$$\mathbf{A}(\mathbf{r},t)=\int\frac{\mathbf{j}\left(\mathbf{r}',t-\left|\mathbf{r}-\mathbf{r}'\right|/c\right)}{\mathbf{r}-\mathbf{r}'}dv \tag{5.36}$$

may be evaluated for sinusoidal waves as,

$$\mathbf{A}(\mathbf{r})=\frac{1}{c}\int\mathbf{j}(\mathbf{r}')\frac{e^{ik(\mathbf{r}-\mathbf{r}')}}{\left|\mathbf{r}-\mathbf{r}'\right|}dv' \tag{5.37}$$

where $k=\omega/c$ is the wave number and the sinusoidal time dependence is assumed. . The term $e^{ik(\mathbf{r}-\mathbf{r}')}/\left|\mathbf{r}-\mathbf{r}'\right|$ can be expanded in terms of $\mathbf{r}$ in spherical coordinates.[4] There are three *lengths* in the integral and

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[4] In all potential problems in which there is an axial symmetry, associated Legendre polynomials and spherical harmonics are used to solve the general purpose boundary value problem. The expansion of the potential $\mathbf{r}$ due to a unit charge at $\mathbf{r}'$ is the most general form Laplace's equation, where $\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-1}^{l}\frac{1}{2l+1}\frac{r^l}{r^{l+1}}\mathbf{Y}_{lm}^*(\theta',\phi')\mathbf{Y}_{lm}(\theta,\phi)$. The spherical harmonics $\mathbf{Y}_{lm}$ are explicitly defined by $\mathbf{Y}_{lm}(\theta,\phi)=N_l^m P_l^m(\cos\theta)e^{im\phi}$, where the associated Legendre polynomial is given by $P_l^{|m|}(\mu)=\left(\left(1/2^l l!\right)(1-\mu^2)^{|m|/2}\right)\left(\left(d^{l+|m|}/d\mu^{l+|m|}\right)\left[(\mu^2-1)^l\right]\right)$ and the normalization factor is given by $\mathbf{N}_l^m=(-1)^{(m+|m|)/2}\left[\left(2l+1/4\pi\right)\left((l-|m|)!/(l+|m|)!\right)\right]^{1/2}$. The peculiar factor $(-1)^{(m+|m|)/2}$ in the

different expansions can be used depending on the relative values of the lengths. These lengths are: the wavelength $\lambda = 2\pi c / \omega$; the distance r to the field point; and the size a of the source. It is always assumed that $\mathbf{r} > \mathbf{r}'$. The relationships between the three *lengths* result in three unique time varying potential equations.

There are also three *regions* in the radiation field that produce different expressions for the time varying potential:

- Near (quasi stationary) $\lambda \ggg r \ggg a$,
- Intermediate (multipole) $\lambda \approx r \ggg a$,
- Far (radiation) $r \ggg \lambda \ggg a$.

The fields have very different properties in the different *zones*. The near zone the fields have the character of static fields with radial components and the variation with distance depends on the properties of the source. In the far zone the fields are transverse to the radius vector and fall off as $1/r$. In the intermediate zone, both dipole and quadrupole approximations are in effect.

§5.4.1. QUASI–STATIONARY EXPANSION

In the near zone, which is call the *Quasi-Stationary* expansion, the relationships between the wavelength, distance from the source and source size are $\lambda \ggg r$ and $\lambda \ggg a$, resulting in a potential equation as a function of the distance, $\mathbf{r}$, of:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}')dv'}{\left|\mathbf{r}-\mathbf{r}'\right|} \equiv \lim_{kr\to 0} \mathbf{A}(\mathbf{r})\frac{1}{c}\sum \frac{4\pi}{2l+1}\frac{\mathbf{Y}_{lm}(\theta)}{r^{l+1}}\int \mathbf{j}(\mathbf{r}')r^{l}\mathbf{Y}^{*}_{lm}(\theta',\phi')dv' . \tag{5.38}$$

The Quasi–Stationary expansion is like the expression for **A** in magnetostatics where the current at a point is stationary, i.e. independent of time. The exponential in Eq. (5.38) can be replaced by unity. Then the vector potential is of the form where the near field oscillates harmonically as $e^{-i\omega t}$, but is otherwise static. [5]

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normalization constant $\mathbf{N}_l^m$ is the conventional phase factor of the spherical harmonic. These phase factors become important when adding linear combinations of spherical harmonics in determining interference effects. [Hobs31], [Arfk85].

[5] Although the Quasi–Stationary approximation derives from assuming that $ik\mathbf{E}_\omega \ll 4\pi\mathbf{j}_\omega / c$ and $ik\mathbf{A}_\omega \ll \nabla\phi_\omega$, i.e. assuming k is small, the approximation cannot be

§5.4.2. MULTIPOLE EXPANSION

In the *Multipole* expansion, the relationships between the wavelength, distance from the source and source size are $\lambda \ggg a$ and $r > a$, resulting in a potential equation as a function of the distance, r, of:

$$\mathbf{A} = \mathbf{A}^{ed} + \mathbf{A}^{md} + \mathbf{A}^{eq}, \tag{5.39}$$

in which the electric dipole potential, $\mathbf{A}^{ed}$ is given by,

$$\mathbf{A}^{ed} = e^{ik\mathbf{r}} \int \mathbf{j}_n dv', \tag{5.40}$$

the magnetic dipole potential, $\mathbf{A}^{md}$ is given by,

$$\mathbf{A}^{md} = \frac{e^{ik\mathbf{r}}}{r}\left(\frac{1}{r} - ik\right)\left(\frac{1}{2c}\int n' \times \mathbf{j}\, dv'\right) \times \mathbf{r}', \tag{5.41}$$

and the electric quadrupole potential $\mathbf{A}^{eq}$ is given by,

$$\mathbf{A}^{eq} = \frac{e^{ik\mathbf{r}}}{2cr}\left(\frac{1}{r} - ik\right)\int\left[\frac{(\mathbf{r}\cdot\mathbf{r}')}{\mathbf{r}}\mathbf{j} + \frac{(\mathbf{r}\cdot\mathbf{j})}{\mathbf{r}}\mathbf{r}'\right]. \tag{5.42}$$

The multipole approximation is used in atomic and nuclear distances, where the conditions for its validity are often satisfied. In the Quasi-Stationary expansion the parameter a/r, the ratio of the source size to the distance to the field point, and the fields of higher multipoles fall off with successfully higher powers of a/r. In the multipole expansion, all multipole fields drop off as $1/r$ in the radiation zone and the parameter that defines the relative importance of successive multipoles is ka not a/r.

§5.4.3. RADIATION EXPANSION

In the far zone or *Radiation* expansion in which $r \ggg a$ and $r \gg \lambda$, the exponential in Eq. (5.38) oscillates rapidly and determines the behavior of the vector potential. In this region $|\mathbf{r} - \mathbf{r}'| \cong r - \mathbf{n}\cdot\mathbf{r}'$ where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$. This results in a potential equation of,

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completely characterized for $k \approx 0$. In the equation $\nabla \times \mathbf{E}_\omega = -ik\mathbf{B}_\omega$, the right hand side cannot be set to zero without reverting to the completely time stationary case for the potential fields.

$$\mathbf{A}(\mathbf{r}) = \frac{e^{ik\mathbf{r}}}{cr}\int \mathbf{j}(r) e^{-ik\mathbf{n}\cdot\mathbf{r}} dv' \equiv \frac{e^{ikr}}{cr}\sum_n \frac{(-ik)^n}{n!}\int \mathbf{j}(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')^n dv' . \qquad (5.43)$$

If only the first expansion terms in kr are considered, than the inverse distance $1/|\mathbf{r}-\mathbf{r}'|$ can be replaced by $1/\mathbf{r}$ to give,

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{r}) = \frac{e^{ikr}}{c\mathbf{r}}\int \mathbf{j}(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} dv \ . \qquad (5.44)$$

The radiation zone expansion is useful in calculating the fields of an arbitrary current distribution at large distances. These far fields fall off as $1/r$, since the original expansion $\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{r\cdot r'}{r^3} + \cdots \approx \frac{1}{r}$, [6] is all that is needed for calculating the rate of radiation energy by means of the Poynting vector. The far zone potential describes an outgoing spherical wave with an angular dependent coefficient. The calculated values from $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = \nabla\times\mathbf{B}$ are transverse to the radius vector and fall off as $1/r$. If the source of the radiation has dimensions small compared to the wavelength, then Eq. (5.44) can be expanded in powers of k to give,

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{r}) = \frac{e^{ikr}}{c\mathbf{r}}\sum_n \frac{(-ik)^n}{n!}\int \mathbf{j}(\mathbf{r})(\mathbf{n}\cdot\mathbf{r}')^n \, dv \ . \qquad (5.45)$$

In the radiation expansion, Eq. (5.40) and Eq. (5.41) and Eq. (5.42) are used to evaluate the time averaged Poynting Vector, such that,

$$\begin{aligned} \overline{S} &= \left[\left(\frac{i}{k}\nabla(\nabla\cdot\mathbf{A}) + ik\mathbf{A}\right)\times(\nabla\times\mathbf{A})\right] + i\left[\left(\frac{i}{k}\nabla(\nabla\cdot\mathbf{A}) + ik\mathbf{A}\right)\times(\nabla\times\mathbf{A})\right], \\ &= \frac{2\omega^4\overline{p}^2}{3c^3}. \end{aligned} \qquad (5.46)$$

---

[6] This expression is derived through a Taylor series expansion of $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$, in increasing powers of the components of $\mathbf{r}'$, such that $\left(\frac{\partial}{\partial x'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)_{r'=0} = -\left(\frac{\partial}{\partial x'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)_{r'=0} = -\frac{\partial}{\partial x}\frac{1}{\mathbf{r}}$, giving, the expansion as,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\mathbf{r}}\left(x'\frac{\partial(1/\mathbf{r})}{\partial x} + y'\frac{\partial(1/\mathbf{r})}{\partial y} + z'\frac{\partial(1/\mathbf{r})}{\partial z}\right) + \frac{1}{2}\left(x'^2\frac{\partial^2(1/\mathbf{r})}{\partial x^2} + y'^2\frac{\partial^2(1/\mathbf{r})}{\partial y^2} + z'^2\frac{\partial(1/\mathbf{r})}{\partial z^2}\right) + \left(x'y'\frac{\partial(1/\mathbf{r})}{\partial x\partial y} + y'z'\frac{\partial(1/\mathbf{r})}{\partial y\partial z} + z'y'\frac{\partial(1/\mathbf{r})}{\partial z\partial y}\right) + \cdots$$

The first expression in brackets decreases like $1/r^2$ with increasing r, the next two terms like $1/r^3$ while further terms decrease with higher powers of r.

where p is the dipole moment of the radiation field. [7]

§5.5. POLARIZATION OF THE RADIATED FIELD

The wave equation developed in the previous section describes the propagation *plane* waves of electromagnetic energy through free space or a conductive media. The concept of *polarization* may be familiar to anyone has worn glasses coated with a *Polaroid* film. This film consists of an array of polymer molecules which has a preference for absorption of light along one specific axis. If two such Polaroid filters are place, one on top of the other, and illuminated from behind, the amount of light passing through the two filters will be a function of how the filters are oriented relative to each other. If the filters are arranged so that the maximum amount of light is transmitted — then one filter of rotated so that the minimum amount of light is transmitted, the intensity of the light passing through the two filters is given by $I = I_0 \cos^2 \theta$, where I is the transmitted intensity, I_0 is the intensity of the light transmitted through the first filter and θ is the angle between the two polarizes. This behavior is Malus' Law, when it was discovered in 1809 by Ètienne Louis Malus (1775–1812) that light becomes partially or completely polarized by reflection [Hall60].

Malus wrote in his memoir in 1809 that...

> ...light reflected by a surface of water at an angle of $52^\circ 45'$ with the vertical, has all the characteristics of one of the beams produced by double refraction ... above and below this angle a part of the ray is more or less modified in a way analogous to that which occurs when light passes through two crystals whose principal sections are neither parallel nor perpendicular [Harm82].

Until the discovery by Faraday in 1846 that a magnetic field can alter the polarization of light, there was little evidence that light and electromagnetism were connected. In 1865 Maxwell published an important paper describing the connection between electromagnetism and light, laying the theoretical groundwork for his *prediction* of electromagnetic radiation [Maxw65].

Since light is an electromagnetic wave the radiation of electromagnetic

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[7] The dipole moment of the radiation field represents the simplest charge distribution, which at large distances $r \gg a$ leads to a field consisting of two charges (poles) of equal strength but of opposite signs. The dipole moment is defined as $p = \int \mathbf{r}' p(\mathbf{r}') dv'$.

waves resulting from Maxwell's equations satisfies the general wave equation,

$$\nabla^2 u - \frac{1}{v}\frac{\partial^2 u}{\partial t^2} = 0, \tag{5.47}$$

where,

$$v = \frac{c}{\sqrt{\mu\varepsilon}}. \tag{5.48}$$

represents the constant velocity of the wave determined by the characteristics of the medium. The Wave Equation in Eq. (5.47) has the general solution,

$$u = e^{ik\cdot\mathbf{r} - i\omega t}, \tag{5.49}$$

where the frequency ω and the magnitude of the propagation constant k are related by,

$$k = \frac{\omega}{v} = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}. \tag{5.50}$$

If waves propagating only in the x–direction are considered, the solution to Eq. (5.50) is,

$$u(x,t) = Ae^{ik\cdot x - i\omega t} + Be^{-ik\cdot x - i\omega t}, \tag{5.51}$$

using Eq. (5.50) gives,

$$u_{\mathbf{k}}(x,t) = Ae^{ik\cdot(x - vt)} + Be^{-ik\cdot(x+vt)}. \tag{5.52}$$

If the velocity v is not a function of the propagation vector k, that is the propagation media is nondispersive, with the permittivity and permeability, $\mu\varepsilon$ independent of frequency the linear superposition of the wave equation results in a general solution of the form,

$$u(\mathbf{r},t) = f(\mathbf{r} - vt) + g(\mathbf{r} + vt), \tag{5.53}$$

where $f(z)$ and $g(z)$ are arbitrary functions. Eq. (5.53) represents waves traveling in the $\mathbf{r}$ direction with phase velocity v.

With the convention that the physical electric and magnetic fields are obtained by taking the *real* parts of the complex quantities, the plane waves take the form of,

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t)&=\mathsf{E}e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t}\\ \mathbf{B}(\mathbf{r},t)&=\mathsf{B}e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t}\end{aligned}\right\},\tag{5.54}$$

where E, B and $\mathbf{n}$ are vectors that are constant in time and space. Each component of the $\mathbf{E}$ and $\mathbf{B}$ field satisfies the wave equation,

$$\nabla^2 u+\frac{\omega^2}{c^2}u=0,\tag{5.55}$$

provided,

$$k^2\mathbf{n}\cdot\mathbf{n}=\frac{\omega^2}{c^2}.\tag{5.56}$$

It is useful at this point to introduce a set of mutually orthogonal unit vectors $(\varepsilon_1,\varepsilon_2,\mathbf{n})$ where $\mathsf{E}=\varepsilon_1 E_0$ and $\mathsf{B}=\varepsilon_2\sqrt{\mu\varepsilon}E_0$. The plane waves in Eq. (5.55) is a wave with its electric field vector in the direction ε_1. Such a wave is *linearly polarized* with polarization vector ε_1. A second wave which linearly independent of the first can be linearly polarized with polarization vector ε_2. The plane waves are now given as,

$$\left.\begin{aligned}\mathbf{E}_1&=\varepsilon_1\mathsf{E}e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t}\\ \mathbf{B}_2&=\varepsilon_2\mathsf{B}e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t}\end{aligned}\right\},\tag{5.57}$$

with,

$$\mathbf{B}_j=\frac{\mathbf{k}\times\mathbf{E}_j}{\mathbf{k}},\quad j=1,2.\tag{5.58}$$

These equations can be combined to describe a homogeneous plane wave propagating in the direction $\mathbf{k}=k\mathbf{n}$, such that,

$$\mathbf{E}(\mathbf{r},t)=(\varepsilon_1 E_1+\varepsilon_2 E_2)\,e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}.\tag{5.59}$$

The amplitudes E_1 and E_2 are complex numbers which allow the description of a phase difference between waves of different polarization. If E_1 *and* E_2 have the *same amplitude* Eq. (5.59) represents a *circularly polarized* wave with its polarization vector having an angle $\theta=\tan^{-1}(E_2/E_1)$ with ε_1 and a magnitude $E=\sqrt{E_1^2+E_2^2}$, as shown Figure 1.0,

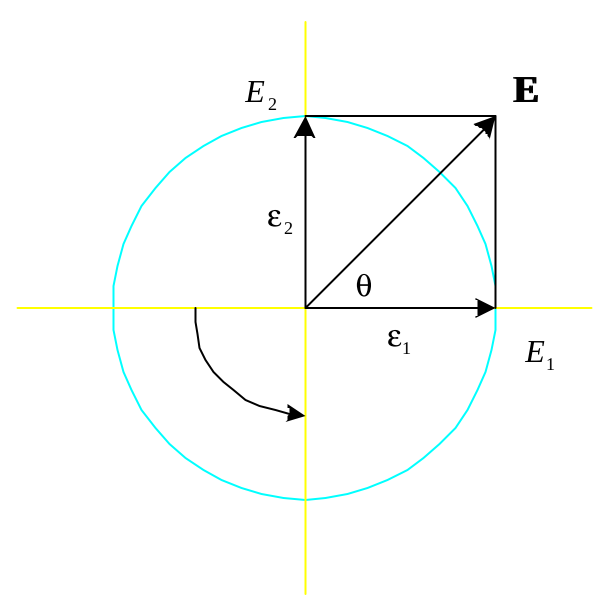

Figure 1.0 — *Circular* Polarization of the Electric Field occurs when a plane wave composed of the components E_1 and E_2 has a constant phase θ and equal amplitudes.

If E_1 *and* E_2 have *different amplitude* Eq. (5.59) represents a *elliptically polarized* wave as shown in Figure 2.0,

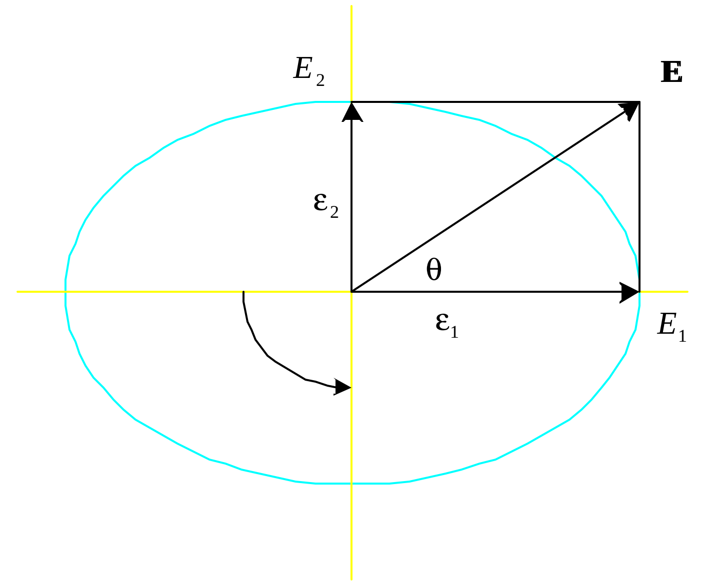

Figure 2.0 — *Elliptical* polarization of the Electric Field occurs when a plane wave composed of the components E_1 and E_2 has a constant phase θ but different amplitudes.

It is the *circularly polarized* wave equation that is of interest in this monograph. To illustrate this, if E_1 and E_2 have the same magnitude, but differ in phase by 90°, the electric field equation is given by,

$$\mathbf{E}(\mathbf{r},t) = E_0 (\varepsilon_1 \pm \varepsilon_2) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{5.60}$$

with E_0 the common *real* amplitude of the propagating waves.

The two *circularly polarized* waves given in Eq. (5.60) form a set of basic fields for the description of the general state of polarization. Further on the this monograph an expression for the *spin* of photons carrying the electromagnetic force will be developed. This concept is directly related to the polarization of propagating electromagnetic waves [Bagg92].

A notation will now be introduced which will later be used in the quantum mechanical description of the photon force particle. By convention, if the electric vector, or the *polarization* vector rotates clockwise when viewed in the direction of propagation, the wave is said to be right circularly polarized. The polarization vector can be given as,

$$\varepsilon_{\mathbf{R}} = -\frac{1}{\sqrt{2}}(\varepsilon_1 + i\varepsilon_2), \tag{5.61}$$

and the left circular polarization vector can be given as,

$$\varepsilon_{\mathbf{L}} = -\frac{1}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2). \tag{5.62}$$

The two complex orthogonal unit vectors in Eq. (5.61) and Eq. (5.62) can be written as,

$$\varepsilon^{(\pm)} = \frac{1}{\sqrt{2}}\left(\varepsilon^{(1)} \pm i\varepsilon^{(2)}\right), \tag{5.63}$$

with,

$$\left.\begin{aligned} \varepsilon^{(\pm)*}\cdot\varepsilon^{(\mp)} &= 0 \\ \varepsilon^{(\pm)*}\cdot\varepsilon^{(\pm)} &= 1 \end{aligned}\right\}. \tag{5.64}$$

The general representation equivalent to Eq. (139) is now given by,

$$\mathbf{E}(\mathbf{r},t) = \left[E^{(+)}\varepsilon^{(+)} + E^{(-)}\varepsilon^{(-)}\right] e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{5.65}$$

where $E^{(+)}$ and $E^{(-)}$ are complex amplitudes.

The direction of rotation is defined as *right–handed* when viewing the wave along its direction of propagation, the electric vector is rotating counter–clockwise. The waves shown in Figure 1.0 and Figure 2.0 are right–handed polarizations. These definitions are used in modern optics and particle physics, the right–handed photon having a *positive* helicity and spin vector in the direction of motion [Dobb85].

> *It has been seen that, on the electromagnetic theory of light, the propagation of waves of light "in vacuo" ought to take place with a velocity equal, within limits of experimental error, to the actual observed velocity of light.*
>
> — Sir James Hopwood Jeans (1877–1946) [Jean25] §589, pp. 532