§17. VACUUM STATE FLUCTUATIONS

Before proceeding with the development of the radiation density of the electromagnetic field, the state containing no photons needs to be addressed further. In order for the *creation* and *annihilation* operators to function they must *operate* on the state of the radiation field, including the vacuum state.

Because to commutator rules given in Eq. (16.10) neither the individual occupation number $N_{\mathbf{k}_i,\alpha_i}$ or the total photon number operator $\mathbf{N} = \sum_{\mathbf{k}} \sum_{\alpha=1,2} N_{\mathbf{k},\alpha} = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \hat{c}^{+}_{\mathbf{k},\alpha} \hat{c}^{-}_{\mathbf{k},\alpha}$ commutes with the $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{A}$ fields. [1]

Like the fundamental commutation relation for a harmonic oscillator, $[q, p] = i\hbar$, which prevents the *simultaneous* vanishing of the potential energy and the kinetic energy, in quantum mechanics *noncommuting* operators cannot be simultaneously determined to arbitrary accuracy's. In the situation where the number of photons is fixed than there are uncertainties in the field strength of the radiation field. This uncertainty follows the *root mean square* measurements developed in the previous section.

As a result the *ground state* of the system has an absolute energy which is non–zero and with variances in the kinetic and potential energies of Δq^2 and Δp^2. This odd behavior can be shown by taking the electric field operator $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and evaluating the expectation value of the field strength. Although the expectation value,

$$\langle 0 | \mathbf{E} | 0 \rangle = 0 \tag{17.1}$$

because,

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[1] The commutators of the field variables $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{A}$ can be derived through the a technique described in [Cohe89]. Let V_m and W_n be two field components. These can be expressed as linear combinations of the annihilation and creation operators $\hat{c}^{+}_{\mathbf{k},\alpha}$ and $\hat{c}_{\mathbf{k},\alpha}$ as, $V_m = \sum_{\mathbf{k}} \sum_{\alpha=1,2} v_{m,\mathbf{k}} \hat{c}_{\mathbf{k},\alpha} + v^{*}_{m,\mathbf{k}} \hat{c}^{+}_{\mathbf{k},\alpha}$ and $W_n = \sum_{\mathbf{k}} \sum_{\alpha=1,2} w_{n,\mathbf{k}} \hat{c}_{\mathbf{k},\alpha} + w^{*}_{n,\mathbf{k}} \hat{c}^{+}_{\mathbf{k},\alpha}$. The commutator is given as, $[V_m, W_n] = \sum_{\mathbf{k}} \sum_{\mathbf{l}} v_{m,\mathbf{k}} w_{m,\mathbf{l}} \left[\hat{c}_{\mathbf{k},\alpha}, \hat{c}_{\mathbf{l},\alpha}\right] + v_{m,\mathbf{k}} w^{*}_{m,\mathbf{l}} \left[\hat{c}_{\mathbf{k},\alpha}, \hat{c}^{+}_{\mathbf{l},\alpha}\right] + v^{*}_{m,\mathbf{k}} w_{m,\mathbf{l}} \left[\hat{c}^{+}_{\mathbf{k},\alpha}, \hat{c}_{\mathbf{l},\alpha}\right] + v^{*}_{m,\mathbf{k}} w^{*}_{m,\mathbf{l}} \left[\hat{c}^{+}_{\mathbf{k},\alpha}, \hat{c}_{\mathbf{l},\alpha}\right]$, which reduces to, $[V_m, W_n] = \sum_{\mathbf{k}} \left(v_{m,\mathbf{k}} w^{*}_{n,\mathbf{k}} - v_{m,\mathbf{k}} w_{n,\mathbf{k}} \right)$.

$$\hat{c}_{\mathbf{k},\alpha}\left|0\right\rangle = 0 , \tag{17.2}$$

the mean square fluctuation of the electric field is given by,

$$\left\langle 0\middle|\mathbf{E}\cdot\mathbf{E}\middle|0\right\rangle - \left|\left\langle 0\middle|\mathbf{E}\middle|0\right\rangle\right|^2 = \left\langle 0\middle|\mathbf{E}\cdot\mathbf{E}\middle|0\right\rangle = \infty . \tag{17.3}$$

The variance of the electric field in the vacuum is then proportional to $\hbar$ and diverges as $\Delta\mathbf{E}^2 = \frac{\hbar}{2\pi^2}\int_0^{k_{\text{maximum}}} k^3 dk$. In a state with definite photon occupation numbers, the electric and magnetic fields are indefinite and fluctuating. The probability distributions for the **E** and **B** fields are analogous to the position and momentum of an oscillator in and energy eigenstate. The result of Eq. (17.3) is that if the occupation number is fixed than the field strength is completely uncertain.[2]

This divergence problem can often be circumvented by recognizing that a practical electromagnetic field is coupled to a material object over a finite bandwidth. The mean square fields associated with the k^{th} photon state averaged over the normalization volume dv and all states up to the k^{th} state gives,

$$\frac{1}{dv}\int\left\langle\left|\mathbf{E}_{\mathbf{k},\alpha}\right|^2\right\rangle dv = \frac{1}{dv}\int\left\langle\left|\mathbf{B}_{\mathbf{k},\alpha}\right|^2\right\rangle dv = \frac{4\pi}{v}\hbar\omega\left(N_{\mathbf{k},\alpha}+\tfrac{1}{2}\right). \tag{17.4}$$

In the practical world the field strength is measured by an instrument of finite dimension and the strength of the field is averaged over a volume defined by $\left\langle\mathbf{E}\right\rangle = \frac{1}{\Delta V}\int_{\Delta V}\mathbf{E}dv$ where ΔV is a small volume containing the measuring device.

The mean square fluctuation can now be given as,

$$\left\langle 0\middle|\left\langle\mathbf{E}\right\rangle\cdot\left\langle\mathbf{E}\right\rangle\middle|0\right\rangle \approx \hbar c\big/\left(\Delta l\right)^4 , \tag{17.5}$$

where Δl is the linear dimension of the volume.

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[2] The infinities of quantum field theory can be dealt with renormalization of the underlying theory, allowing the infinities to be ignored. P. A. M. Dirac [Dira58] commented on the renormalization by saying ... *the rules ... do not fit in with the logical foundations of quantum mechanics. They showed therefore not be considered as a satisfactory solution to the difficulties.* In the final sentence Dirac states, *The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics.*

§17.1 RADIATION DENSITY OF THE QUANTIZED FIELD

All the peculiarities of the quantized radiation field must somehow be brought back to the classical description. The properties of a quantum electromagnetic field are similar to the classical electromagnetic field properties when the quantum numbers, $N_{\mathbf{k},\alpha}$, defining the stationary states of the field oscillators, are large.

> *In nature's infinite book of secrecy*
>
> *A little I can read*
>
> — W. Shakespeare

The total field energy, $\langle \mathbf{E} \rangle$, per unit volume is then proportional to the number of photons is state $\mathbf{k}$, such that,

$$N_{\mathbf{k}} \approx \langle \mathbf{E} \rangle^2 c^3 \big/ / \hbar\omega^4 . \tag{17.6}$$

With this number is large, the field energy, $\bar{\mathbf{E}}$, is,

$$\langle \mathbf{E} \rangle \ggg \sqrt{\hbar c} \big/ (c\,\Delta t)^2 , \tag{17.7}$$

which allows the field averaged over time intervals Δt to be treated as classical.

The field strength for a classical electromagnetic wave of wavelength $2\pi\lambda$ is comparable to the quantum field operator $\hat{\mathbf{E}}$, such that,

$$\hat{\mathbf{E}}^2 \approx \hbar c \big/ \lambda^4 , \tag{17.8}$$

where the average is taken over the volume λ^3. The time average of $\hat{\mathbf{E}}^2$ can be equated to the energy density of the electromagnetic wave, so that,

$$\left\langle \hat{\mathbf{E}}^2 \right\rangle = \bar{n}\hbar \frac{c}{\lambda}, \tag{17.9}$$

where $\bar{n}$ is the number of photons per unit volume.

In order to observe the classical behavior of the radiation field, the quantum mechanical effects must be negligible for large numbers of photons, such that,

$$\bar{n} \ggg \frac{1}{\lambda^3}, \tag{17.10}$$

resulting in the description of physical phenomena based on classical electrodynamics when the number of photons per unit volume is much greater than one.

For a radiation field whose source is generating a 100 MHz ($\lambda \approx 48\text{cm}$), with a power of 140,000 watts, the number of photons per unit volume at a distance of 5.0 miles from the transmitter is on the order of 1.0×10^{17}. In order to calculate the number of photons at a distance from the radiation source, the ratio the volume containing the photons to the overall volume of the radiation field will be used to adjust the total energy density. **Figure 5.0** describes the volume element of interest.

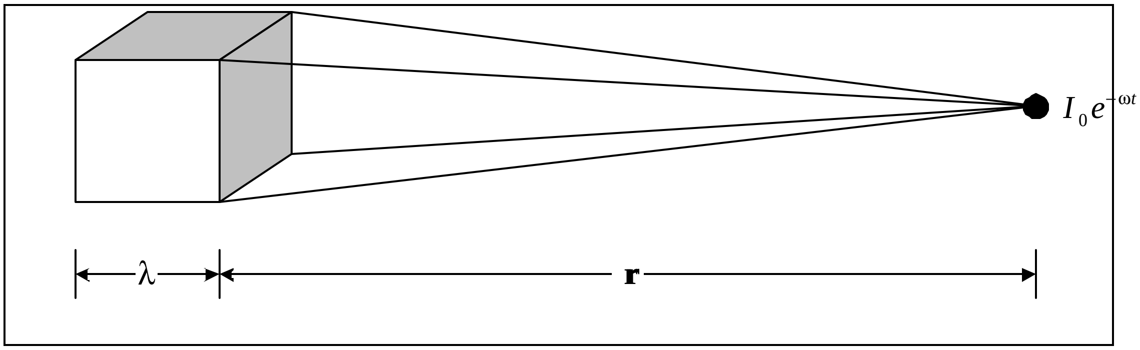

Figure 5.0 — Volume element of the radiation field density containing a calculated number of photons *radiating* from the radio antenna distance *r* away. In order to maintain the correspondence principle, the number of photons in this *unit* volume must be *very large* compared to the quantum mechanical description of the electromagnetic field.

The volume of a sphere [Owen61], in spherical coordinates is given by,

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 dr \sin\theta d\theta d\phi = \frac{4}{3}\pi R^3 . \qquad (17.11)$$

For a unit sphere of radius $R = 1$, the volume is $V \cong 2.819 \text{ cm}^3$. For a sphere of radius 5 miles (160,934 cm), the volume is $V \cong 1.7416 \times 10^{16} \text{ cm}^3$. The ratio of the total volume of the radiation sphere to the small volume area of interest is,

$$ratio = \frac{1.106 \times 10^5 cm^3}{1.7416 \times 10^{16} cm^3} = 6.35 \times 10^{-12} . \qquad (17.12)$$

This ratio will be used to adjust the total number of photons to the number contained in the volume of interest.

Solving for the total number of photons $\bar{n}$ radiated by the transmitter gives,

$$\begin{aligned}\bar{n} = \frac{\langle E^2 \rangle \lambda}{\hbar c} &= \frac{\left(1\times10^5\,\mathrm{W}\right)\left(1\times10^7\,\mathrm{erg/sec}\right)\left(48\mathrm{cm}\right)}{\left(1.05\times10^{-25}\mathrm{erg}\right)\left(1\times10^{10}\mathrm{cm/sec}\right)} \\ &= \frac{\left(4.8\times10^{13}\,\mathrm{erg\ cm/sec}\right)}{\left(1.05\times10^{-15}\mathrm{erg\ cm/sec}\right)} = 4.57\times10^{28}\end{aligned} \tag{17.13}$$

Adjusting the total number of photons in the spherical volume for the number of photons in the volume of interest gives,

$$\bar{n} = 4.57\times 10^{28} \times 6.35\times10^{-12} \cong 1.0\times10^{17}\,. \tag{17.14}$$

§17.2 RADIATION DAMPING AND SELF FIELDS

Now that the quantum field description of the electromagnetic field has be shown to be *extendible* to the classical radiation field, the effect on the electrons in the receiving antenna will be examined from the point of view of quantum field theory.

In the description of the classical equations of motion of electrons in a conducting material the problem of electrodynamics could be divided into two categories. The first in which the external electromagnetic field is applied to charges — the receiving antenna — and their resulting motions calculated and the second where the motion of charges resulted in an electromagnetic field — the transmitting antenna. The description produced in both situations only approximate the actual behavior of nature.

In the case where an external field is incident on a charge, the motion of the charges involve the emission of radiation. This radiation carries off energy, momentum and angular momentum and therefore influences the subsequent motion of the charged particles. In the macro–world these effects cause negligible errors in the calculations of measurable quantities. Although these solutions are *workable* the basic problem remains unsolved for the micro–world description of the electrons motion.

If an external field causes a particle of charge e to have an acceleration a for a period t then the radiated energy is approximately,

$$E_{\mathrm{rad}} \sim \frac{2e^2 a^2 t}{3c^2}, \tag{17.15}$$

which is Larmor's formula.[3] If this radiated energy is very small compared to the incident energy E_0 then the radiate effects will be unimportant.

If however the radiated energy is approximately equal to the incident energy, that is $E_{\text{rad}} \approxeq E_0$, the effects of the radiative reaction on the motion of the charged particle will be appreciable. This relationship between the incident energy E_0 and the radiated energy can be defined in one of two ways. If a charged particle is initially at rest and a force is applied to it for a limited time t, than the particle will be accelerated continuously for that time period. The relevant energy can then be given as,

$$E_0 \sim m(at)^2 . \tag{17.16}$$

The determination of whether the radiative effects are important becomes,

$$\frac{2e^2a^2t}{3c^2} \ll ma^2t , \tag{17.17}$$

or,

$$\tau \equiv t \gg \frac{2e^2}{3mc^2} , \tag{17.18}$$

which is called the *characteristic time* of the interaction. The t is long compared to τ then the radiation corrections are unimportant. For electrons the characteristic time $\tau = 6.26\times10^{-24}\,\text{sec}$. This can be restated as a distance, since light will travel 10^{-13}cm in this time. The radiative corrections are unimportant if the interaction is less than this distance.

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[3] Larmor's formula describes the total radiated power produced by an accelerated charge [JACK75]. This expression can be derived starting from Poynting's vector, $\overline{S} = \mathbf{E}\times\mathbf{B} = c/4\pi|\mathbf{E}|^2\cdot\mathbf{n}$. The power radiated per unit of solid angle is given as, $dP/d\Omega = \left(e^2/4\pi c\right)\left|\mathbf{n}\times\left(\mathbf{n}\times\dot{\beta}\right)\right|^2$ where $\beta = v(t)/c$ is the distance traveled by the charged particle. If Θ is the angle between the acceleration vector $\dot{v}$ and $\mathbf{n}$ then the radiated power is, $dP/d\Omega = \left(e^2/4\pi c\right)|\dot{v}|^2\sin^2\Theta$. The total instantaneous power radiated by the accelerating particle can be found by integrating over all solid angles to give, $P = \left(2e^2/3c^3\right)|\dot{v}|^2$.

For a charged particle undergoing a periodic motion at a characteristic frequency of ω_0 the mechanical energy associated with the particles motion can be identified with E_0 and is given by,

$$E_0 \sim m\omega_0^2 d^2 , \tag{17.19}$$

where the accelerations are typically $a \sim \omega_0^2 d$ and the time intervals are $t \sim 1/\omega_0$. The determination of the radiative effects are now given as,

$$\frac{2e^2\omega_0^2 d^2}{3c^2\omega_0} \ll m\omega_0^2 d , \tag{17.20}$$

or

$$\omega_0 \tau \ll 1 . \tag{17.21}$$

where τ is again the characteristic time. Since $1/\omega_0$ is a time associated with the mechanical motion, if this inverse *frequency* or *time* is long compared the characteristic time than the radiative effects or unimportant.

For classical electrodynamics times of τ or distances of $c\tau$ are sufficiently small so that the radiative effects can be ignored.

§17.3 OPEN QUESTIONS ABOUT THE QFT

In the previous sections the number of photons in a volume was given by a *simple expression* which was derived from the average energy density of the quantized radiation field. This number was introduced after a lengthy background development which included many diversions. The result of the arduous journey is a surprisingly simple expression, $\left\langle \hat{\mathbf{E}}^2 \right\rangle = \bar{n}\hbar\frac{c}{\lambda}$. As usually is the case in a subject like this, the final result is a bit disappointing.

The path to this conclusion started with Maxwell's classical fields equations, which originally depended on a propagation media — the ether. These fields then became a material object themselves through the development of *action-by-contact*. Quantizing the field produced a *coupled oscillator* description of the radiation. Through the mathematics of the oscillator, particles carrying the force were introduced — making the transition to quantum field theory and the beginnings of quantum electrodynamics.

All that really happened — in the end — is that the quantum mechanical description of the electromagnetic radiation field has been verified to produce the proper result when compared to the classical field equation description — *a restatement of the obvious.*

But there is more than meets the eye here:

- The transition to a particle based description of the radiation field has been made, laying the groundwork for *Quantum Electrodynamics*.
- The details of the potential field Hamiltonian and gauge invariance have been explored.
- The background has been developed to ask the original question in light of a new understanding.

WHAT CAUSES THE ELECTRONS IN THE RECEIVING ANTENNA TO MOVE?

The answer to this question seems no more plausible than the classical answer — a force causes the electrons to move/ The actual cause is just as mysterious as before:

- Are there photons *flying* across empty space and colliding with the electrons in the antenna?
- Are there actual photons be emitted by the accelerated electrons in the transmitting antenna?

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The quantum explanation uses the photon as the gauge particle because the mathematics allows it to do so — not because they are physically there.

And in the end, when man has fully partaken of the fruit of the tree of knowledge, there will be this difference between the first Eden and the last, that man will not become as a god, but remain forever humble.

— P. W. Bridgeman [Brid29]

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